The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science: Series 7
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/tphm18

XXX. Theory of ionization fluctuations

J.E. Moyal [a] [b]
[a] Department of Mathematics , The University , Manchester
[b] F.B.S. Falkiner Nuclear Research and Adolph Basser Computing Laboratories, School of Physics , The University of Sydney , Sydney , Australia
Published online: 22 Apr 2009.

XXX. *Theory of Ionization Fluctuations*

By J. E. Moyal*
Department of Mathematics, The University, Manchester†

[Received August 20, 1954]

Summary

The distributions of the loss of energy by ionization of a fast primary and of the numbers of ion pairs it produces are derived. It is shown that down to quite small values of the primary ionization, both can be represented by the same 'universal' distribution if the variables are reduced by a proper choice of scale and origin, which accounts for the experimental fact that ion pair numbers are proportional to primary energy loss. These conclusions remain valid when one takes into account quantum resonance effects and the details of atomic structure of the absorber.

§ 1. Introduction

The main object of the present paper is to derive expressions for the distributions of (*a*) the loss of energy by ionization of a fast primary particle passing through an absorbing medium; (*b*) the numbers of ion-pairs produced by such a particle. We shall try in particular to explain the experimental fact that these two distributions are very approximately proportional to each other: i.e. that the energy lost by the primary particle per ion-pair produced is approximately constant (of the order of 35 ev).

Landau (1944) has shown that under certain simplifying assumptions, the distribution of energy loss can be expressed as a universal curve in terms of certain reduced energy variables (depending on the charge, mass and velocity of the primary, the atomic properties and the thickness of the absorber). We shall first derive a closed analytic expression for Landau's distribution. In view of discrepancies between this distribution and the results of recent experimental work (see West 1953) we shall examine possible departures from it due to: (*a*) small thicknesses of absorber; (*b*) the influence of the detailed atomic structure of the absorber; (*c*) the influence of quantum resonance effects in distant collisions. Finally, we shall derive a theoretical expression for the distribution of the numbers of ion pairs produced by the primary. This will allow us to decide whether these discrepancies can be due to the usual

* Communicated by G. D. Rochester.

† At present at the F.B.S. Falkiner Nuclear Research and Adolph Basser Computing Laboratories, School of Physics, The University of Sydney, Sydney, Australia.

assumption made in cloud chamber, ionization chamber and proportional counter work of a constant energy loss per ion-pair. The experimental evidence will be analysed in a forthcoming paper by Owen and Eyeions; a review of the subject by Price will appear in *Reports on Progress in Physics*.

§ 2. General Expression for the Ionization Energy Distribution

In deriving an expression for the ionization energy distribution, we shall make the following assumptions:

(1) Successive ionizing collisions are statistically independent.

(2) The total energy loss of the primary is very much smaller than its initial energy, and hence the decrease of primary energy may be neglected. Our theory will not therefore apply to slow primaries or great thicknesses of absorber. We also assume that energy losses due to radiation or nuclear interactions are negligible.

(3) The absorbing medium has a homogeneous constitution.

We shall develop the theory first for an arbitrary total cross section $\sigma(E)$, where $N\sigma(E)\,dE\,dt$ is the probability of an energy loss between E and $E+dE$ in an absorber thickness dt, N being the number of absorber atoms per unit volume. The *primary ionization rate* is then

$$q=N\int_0^\infty \sigma(E)\,dE,$$

and the ionization energy distribution per collision is

$$\phi(E)=\frac{N\sigma(E)}{q}, \qquad \text{with} \qquad \int_0^\infty \phi(E)\,dE=1$$

($q\,dt$ is the probability of an ionizing collision in the thickness dt, $\phi(E)\,dE$ the probability of an energy loss between E and $E+dE$ *given* that a collision has occurred).

Let $\chi(E,t)\,dE$ be the probability of an energy loss between E and $E+dE$ in a thickness t. It is easily seen to follow from assumptions (2.1), (2.2) and (2.3) that

$$\chi(E,t_1+t_2)=\int_0^E \chi(E-W,t_1)\chi(W,t_2)\,dW. \quad . \quad . \quad . \quad (2.1)$$

For a small thickness δt, the probability that no energy is lost is $1-q\delta t+o(\delta t)$; the probability of an energy loss between E and $E+dE$ is $q\phi(E)\,dE\delta t+o(\delta t)$; hence

$$\chi(E,\delta t)=(1-q\delta t)\delta(E)+q\phi(E)\delta t+o(\delta t); \quad . \quad . \quad . \quad (2.2)$$

in particular, $\chi(E,0)=\delta(E)$.*

We introduce the Laplace transform of $\chi(E,t)$:

$$M(\alpha,t)=\int_0^\infty \exp(-\alpha E)\chi(E,t)\,dE. \quad . \quad . \quad . \quad . \quad (2.3)$$

* The Dirac δ-function $\delta(E)$ should not be confused with δt.

It follows from (2.1) that

$$M(\alpha, t_1+t_2)=M(\alpha, t_1)M(\alpha, t_2), \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.4)$$

and hence from (2.2) and (2.4) (noting that $M(\alpha, 0)=1$) that

$$\begin{aligned}\frac{\partial M(\alpha, t)}{\partial t} &= \lim_{\delta t\to 0}\frac{1}{\delta t}\{M(\alpha, \delta t)-1\}M(\alpha, t)\\ &= \lim_{\delta t\to 0}\frac{1}{\delta t}\left\{(1-q\delta t)+q\delta t\int_0^\infty \exp(-\alpha E)\phi(E)\,dE+o(\delta t)-1\right\}M(\alpha, t)\\ &= \left\{q\int_0^\infty [\exp(-\alpha E)-1]\phi(E)\,dE\right\}M(\alpha, t). \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.5)\end{aligned}$$

The solution of the differential eqn. (2.5) with initial condition $M(\alpha, 0)=1$ is

$$M(\alpha, t)=\exp\left\{qt\int_0^\infty [\exp(-\alpha E)-1]\phi(E)\,dE\right\}=\exp[QR(\alpha)], \quad (2.6)$$

where $Q=qt$ is the *mean number of collisions* (i.e. the *primary ionization*) in the thickness t, and

$$R(\alpha)=\int_0^\infty [\exp(-\alpha E)-1]\phi(E)\,dE. \quad . \quad . \quad . \quad . \quad . \quad (2.7)$$

The standard inversion formula for Laplace transforms then gives

$$\chi(E, Q)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\exp[QR(z)+zE]\,dz \quad . \quad . \quad . \quad (2.8)$$

(it is convenient to express χ in terms of the mean collision number Q instead of the thickness t).

An asymptotic expansion for χ may be found from (2.8) by the saddle point method (see e.g. Jeffreys 1946 and Daniels 1954). As is usual in applications of this method, we can accept as a sufficient approximation to χ the first term of this expansion, namely, the expression

$$\chi(E, Q)\,dE=\frac{1}{c}[2\pi QR''(\alpha)]^{-1/2}\exp\{Q[R(\alpha)-\alpha R'(\alpha)]\}\,dE, \quad (2.9)$$

where α is related to E by the expression

$$E=-QR'(\alpha)=Q\int_0^\infty E\exp(-\alpha E)\phi(E)\,dE, \quad . \quad . \quad (2.10)$$

$$R''(\alpha)=\int_0^\infty E^2\exp(-\alpha E)\phi(E)\,dE, \quad . \quad . \quad . \quad . \quad . \quad (2.11)$$

and c is a normalization constant, chosen to make

$$\int_0^\infty \chi(E, Q)\,dE=1.$$

The most probable energy loss E_p is by definition the value of E for which $\chi(E, Q)$ is maximum. Maximizing the right-hand side of (2.9) with respect to α, we see that $E_p=-QR'(\alpha_p)$, where α_p is the solution of

$$R'''(\alpha_p)+2Q\alpha_p[R''(\alpha_p)]^2=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.12)$$

§ 3. The Classical (Thomson) Cross Section

Following Landau (*loc. cit.*) we now take for $\sigma(E)$ the classical (Thomson) cross section (see e.g. Bohr 1948)

$$N\sigma(E)\,dE=\begin{cases} BZ\,dE/E^2 & \text{for} \quad E_0\leqslant E\leqslant E_m, \\ 0 & \ldots \quad E<E_0 \quad \text{or} \quad E>E_m. \end{cases} \quad . \quad (3.1)$$

where

$$B=2\pi N\frac{z^2\epsilon^4}{\mu v^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.2)$$

N is the number of atoms per unit volume in the absorber, μ the mass of the electron, ϵ its charge, $z\epsilon$ the charge of the primary, v its velocity, Z the 'effective' atomic number of the absorber; E_0 and E_m represent average values for respectively the minimum and maximum transferable energy per electron in an ionizing collision (see § 5). It follows that

$$q=BZ\int_{E_0}^{E_m}\frac{dE}{E^2}=BZ\left(\frac{1}{E_0}-\frac{1}{E_m}\right)\simeq\frac{BZ}{E_0} \quad \text{and hence} \quad Q\simeq\frac{BZt}{E_0}.$$

$$\phi(E)\,dE=E_0\frac{dE}{E^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.3)$$

This cross section gives the classical expression for the average energy loss in thickness t

$$\bar{E}=Nt\int_0^\infty E\sigma(E)\,dE=BZt\int_{E_0}^{E_m}\frac{dE}{E}=BZt\log\frac{E_m}{E_0}, \quad . \quad . \quad . \quad (3.4)$$

known to be too small by approximately a factor of 2. The integral diverges if we make $E_m\to\infty$. However, it is found that for fast primaries the value of E_m has a negligible effect on the shape of the energy loss distribution curve;* this renders legitimate the simplifying assumption that E_m is infinite.

It is convenient when substituting from (3.1) into the expressions of § 2 to change to the energy variable $\epsilon=E/E_0$; we then find that

$$R(\alpha)=\exp(-\alpha)-1-\alpha\int_1^\infty\exp[-\alpha\epsilon](d\epsilon/\epsilon);$$

$$R'(\alpha)=-\int_1^\alpha\exp(-\alpha\epsilon)\frac{d\epsilon}{\epsilon}; \quad R''(\alpha)=\frac{e^{-\alpha}}{\alpha}; \quad R'''(\alpha)=-\frac{e^{-\alpha}}{\alpha}\left(1+\frac{1}{\alpha}\right). \quad . \quad . \quad . \quad . \quad (3.5)$$

Hence

$$\chi(\epsilon,Q)\,d\epsilon=\frac{1}{c}\sqrt{\left(\frac{\alpha}{2\pi Q}\right)}\exp\left[\tfrac{1}{2}\alpha+Q(e^{-\alpha}-1)\right]d\epsilon, \quad . \quad . \quad (3.6)$$

where

$$c=\sqrt{\left(\frac{Q}{2\pi}\right)}\int_0^\alpha\exp\left[-\tfrac{1}{2}\alpha+Q(e^{-\alpha}-1)\right]\frac{d\alpha}{\sqrt{\alpha}}$$

$$=\exp(-Q)\sum_{n=0}^\infty\frac{1}{\sqrt{(2n+1)}\,n!}Q^{n+1/2}. \quad . \quad . \quad . \quad . \quad (3.7)$$

---

* This curve (see fig. 3) consists of a sharp peak centered on E_p (representing the statistical effect of frequent collisions with small energy transfer) followed by a long tail of small ordinate (representing the effect of rare violent collisions). Taking the finiteness of E_m into account merely alters this tail in such a way as to make $\bar{E}<\infty$ without appreciably affecting the main part of the curve.

The integral above is taken from 0 to ∞ because α decreases from ∞ to 0 as ϵ increases from 0 to ∞. This can be seen from relation (2.9) which connects ϵ to α:

$$\frac{\epsilon}{Q}=-R'(\alpha)=\int_{\alpha}^{\infty} e^{-x}\frac{dx}{x}=-\mathrm{Ei}(-\alpha)\,; \quad . \quad . \quad . \quad . \quad (3.8)$$

the exponential integral $\mathrm{Ei}(x)$ is tabulated (cf. Janke-Emde 1938 or *British Association Tables* 1951). For small values of α, however, ($\alpha\leqslant 1$), it is more convenient for computation purposes to transform (3.8) as follows:

$$\frac{\epsilon}{Q}=\int_{1}^{\infty} e^{-x}\frac{dx}{x}-\int_{0}^{1}(e^{-x}-1)\frac{dx}{x}+\int_{0}^{\alpha}(e^{-x}-1)\frac{dx}{x}-\log\alpha$$

$$=-C-\log\alpha+\int_{0}^{\alpha}(e^{-x}-1)\frac{dx}{x}$$

$$=-C-\log\alpha-\sum_{n=1}^{\infty}(-1)^{n}\frac{\alpha^{n}}{n\,.\,n\,!}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.8\,a)$$

where $C=0{\cdot}577$ is Euler's constant. The numerical evaluation of χ is discussed in the next section in terms of corrections for small Q to Landau's universal distribution.

Substituting from the above in (2.12), we find the value of α_p of α which maximizes (3.6) by solving

$$2Q\exp(-\alpha_p)=1+\frac{1}{\alpha_p}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.9)$$

graphically (fig. 1), and the corresponding most probable value of the energy loss ϵ_p from (3.8): ϵ_p is shown as a function of Q in fig. 2. It is seen that the equation has no solution for $Q<2{\cdot}44$, indicating that for such low values of Q, $\chi(\epsilon,Q)$ does not go through a maximum, but decreases more or less exponentially as ϵ increases. However, the accuracy of the saddle-point approximation is poor for such low Q; for $Q>5$, we see that $\alpha_p<0{\cdot}13$: hence we can approximate to (3.9) by the equation

$$\frac{1}{2Q}\,\frac{1}{\alpha_p}\left(1+\frac{1}{\alpha_p}\right)=\frac{1}{\alpha_p}-1, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.10)$$

whose solution is to a good approximation

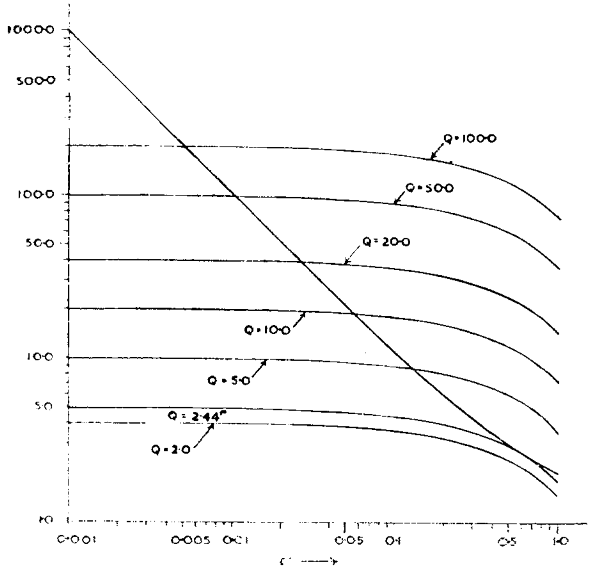
$$\frac{1}{\alpha_p}=2Q-2-\frac{1}{2Q-1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.11)$$

Taking the first three terms in the right-hand side of (3.9) we obtain for the most probable energy loss the approximate expression

$$\epsilon_p=\frac{E_p}{E_0}=Q\left\{\log\left(2Q-2-\frac{1}{2Q-1}\right)-C+\frac{2Q-1}{(2Q-2)(2Q-1)-1}\right\}, \quad (3.12)$$

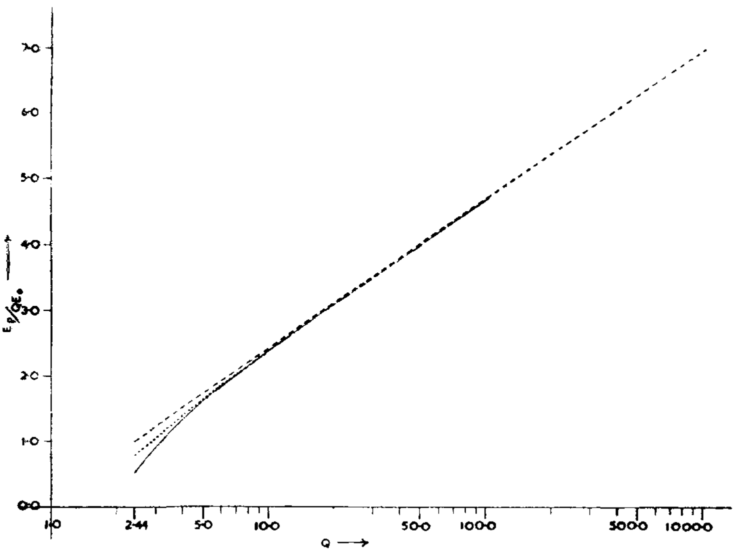
which for $Q\geqslant 5$ departs by less than 1% from the value obtained by the graphical solution of (3.9) (cf. fig. 2).

Fig. 1



Solution of eqn. (3.9) : the values of α_p are given by the abscissae of the intersections of the curve $1+1/\alpha$ with the curves $2Q\,e^{-\alpha}$ for values of $Q=2{\cdot}44$, 5, 10, 20, 50, 100 ; there is no intersection (and hence no maximum) for $Q<2{\cdot}44$.

Fig. 2



The most probable energy loss E_p (in units QE_0) as a function of the primary ionization Q : the full curve is obtained by numerical solution of eqn. (3.9), the dotted and dashed curves represent respectively the approximations of eqn. (3.12), and (4.4).

§ 4. The Landau Approximation

One finds that for large Q, the major contribution to $\chi(\epsilon, Q)$ in (3.6) comes from small values of α (as we have seen, α_p is small for $Q \geqslant 5$, and the curve for χ is sharply centred about ϵ_p). Hence, following Landau, we obtain an asymptotic expression for χ valid for large Q by neglecting terms of order α compared to $\log \alpha$: thus, we take

$$R(\alpha) = \alpha(C - 1 + \log \alpha). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.1)$$

Substituting these values, (3.6) and (3.8 a) become

$$\chi(\epsilon, Q)\, d\epsilon = \frac{1}{c} \sqrt{\left(\frac{\alpha}{2\pi Q}\right)} \exp(-Q\alpha)\, d\epsilon,$$

where

$$c = \frac{1}{\sqrt{(2\pi)}} \int_0^\infty \exp(-Q\alpha) \frac{Q\,d\alpha}{\sqrt{(Q\alpha)}} = \frac{1}{\sqrt{2}}, \quad . \quad . \quad . \quad (4.2)$$

$$\frac{\epsilon}{Q} = -C - \log \alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.3)$$

To the same order of approximation, we may take for the most probable values

$$\alpha_p' = \frac{1}{2Q} \quad \text{and} \quad \frac{\epsilon_p'}{Q} = \frac{E_p'}{BZt} = -C + \log 2Q. \quad . \quad . \quad (4.4)$$

Changing to the *reduced energy variable*

$$\omega = \frac{\epsilon - \epsilon_p'}{Q} = \frac{E - E_p'}{BZt} = -\log 2Q\alpha, \quad . \quad . \quad . \quad . \quad (4.5)$$

we find the following explicit expression for Landau's distribution

$$\chi_L(\omega)\, d\omega = \frac{1}{\sqrt{(2\pi)}} \exp\{-\tfrac{1}{2}[\omega + \exp(-\omega)]\}\, d\omega. \quad . \quad . \quad (4.6)$$

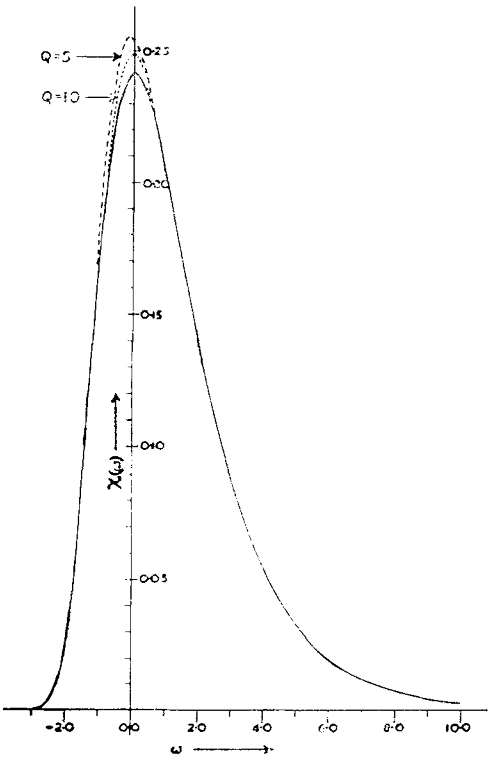
The *half-width* $\Delta\omega$ of this distribution is easily found to be $\Delta\omega = 3{\cdot}58$.

We see thus that the asymptotic expression for χ has a 'universal' form independent of Q when the energy is expressed in terms of the reduced variable ω. The accuracy of this expression has been assessed by computing χ for small values of Q from the more accurate expression (3.6) and changing over to the reduced variable (4.5): the results are shown in fig. 3. Surprisingly, *no appreciable departure is found from Landau's distribution for* $Q \geqslant 20$; the departure for $Q = 10$ and $Q = 5$ is indicated on the figure by dotted lines. We must mention that the expressions for Q and E_p above are not the same as Landau's: the relativistic log rise with increasing energy is absent, because we have neglected quantum resonance effects (see the discussion at the end of § 6).

§ 5. Effects of Atomic Structure

In order to assess the detailed effects of the atomic structure of the absorber on the distribution of ionization energy, we shall use a crude classical model: we assume that the classical cross section is valid for the electrons in each shell of the atom, with a minimum transferable energy

Fig. 3



Distribution of primary energy loss: the full curve is Landau's 'universal' distribution (4.6), the dashed curve is computed from eqns. (3.6) and (3.8), for $Q=5$ and 10.

equal to the ionization potential of the shell. Let I_j be the ionization potential of the jth electron, $j=1, 2, \ldots, Z$; then the total cross section is

$$\sigma(E)\,dE=\sum_{j=1}^{Z}\sigma_j(E)\,dE, \text{ with } N\sigma_j(E)=\begin{cases}B/E^2 & \text{for } E\geqslant I_j, \\ 0 & \text{,, } E<I_j.\end{cases} \quad (5.1)$$

where B has the same definition as in eqn. (3.2). Then

$$Q=Nt\int_0^\infty \sigma(E)\,dE=Bt\sum_{j=1}^{Z}\int_{I_j}^{\infty}\frac{dE}{E^2}=Bt\sum_{j=1}^{Z}\frac{1}{I_j}=\frac{BZt}{E_a}, \quad (5.2)$$

where

$$\frac{Z}{E_a}=\sum_{j=1}^{Z}\frac{1}{I_j} \quad (5.3)$$

and

$$\phi(E)\,dE=\frac{E_a}{BZ}N\sum_{j=1}^{Z}\sigma_j(E)\,dE. \quad (5.4)$$

Changing to the energy variable $\epsilon=E/E_a$, and substituting in the expressions of § 2, we find $R(\alpha)$ and hence $\chi(\epsilon, Q)$. It turns out that an asymptotic expression for χ similar to that derived in § 4 is valid for values of $Q \geqslant 20$. Restricting our attention to this case, we find that to the same order of approximation as in § 4,

$$R(\alpha)=\alpha\left[C+1-\log\left(\frac{\alpha E_b}{E_a}\right)\right], \quad . \quad . \quad . \quad . \quad . \quad (5.5)$$

where $\log E_b=(1/Z)\sum_{j=1}^{Z}\log I_j$. (Note that $\bar{E}=BZt\log(E_m/E_b)$; i.e. in the present classical model, E_a and E_b are the average ionization potentials appropriate for the calculation of respectively the primary ionization Q and the average energy loss $\bar{E}$.) We then find for the most probable values $\alpha_p'=1/2Q$, and hence

$$\frac{\epsilon_p'}{Q}=\frac{E_p'}{BZt}=-C+\log\frac{2QE_a}{E_b}$$

$$=-C+\log\frac{2BZt}{E_b}. \quad . \quad . \quad . \quad . \quad (5.6)$$

Introducing the reduced energy variable

$$\omega=\frac{\epsilon-\epsilon_p'}{Q}=\frac{E-E_p'}{BZt}=-\log 2Q\alpha, \quad . \quad . \quad . \quad . \quad (5.7)$$

we see at once that the asymptotic distribution is exactly the same as Landau's, namely

$$\chi_L(\omega)\,d\omega=\frac{1}{\sqrt{(2\pi)}}\exp\{-\tfrac{1}{2}[\omega+\exp(-\omega)]\}\,d\omega. \quad . \quad . \quad (5.8)$$

Thus, the details of atomic structure, with the assumptions made above, have no effect on the asymptotic distribution* but only modify the expression for the average number of collisions Q and the most probable energy loss E_p: these conclusions are not likely to be modified by any more refined theory of these effects, though we may expect improvements in the expressions for Q and E_p.

§ 6. Quantum Resonance Effects

The classical cross section proportional to $1/E^2$ is valid for values of $E \gg I$, the average ionization potential of the absorber. Its failure in case $\epsilon^2/\hbar v=c/137v \ll 1$ (where $\hbar$ is $\frac{1}{2}\pi$ times Planck's constant) may be ascribed to quantum resonance effects in distant collisions, which increase the cross section for values of E near I, and consequently approximately double the mean energy loss (see e.g. Bohr, *loc. cit.*, p. 89 for a discussion). In order to assess the effect of this resonant increase on the ionization

* Blunk *et al.* (1950, 1951) appear to arrive at a contrary conclusion.

energy distribution, we shall postulate (since an exact expression is not available) a cross section

$$\sigma(E)\, dE = \frac{BZ}{N}\, \frac{dE}{(E-I)^2+\Gamma^2} \cdot \qquad . \quad . \quad . \quad . \quad . \quad (6.1)$$

of the usual form for quantum resonance effects, Γ being the half-width of the resonance curve. We expect that $\Gamma < I$; hence σ is approximately equal to the classical cross section when $E \gg I$. In calculating the ionization energy distribution, we must set $\sigma(E)=0$ for non-ionizing collisions where $E<I$. Hence

$$\left.\begin{aligned} Q &= Nt\int_I^\infty \sigma(E)\, dE = BZt\int_I^\infty \frac{dE}{(E-I)^2+\Gamma^2} = BZt\int_0^\infty \frac{dx}{x^2+\Gamma^2} = \frac{\pi}{2}\,\frac{BZt}{\Gamma}, \\ \phi(E)\, dE &= \frac{2\Gamma}{\pi}\,\frac{dE}{(E-I)^2+\Gamma^2} \cdot \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (6.2)$$

Transforming to the energy variable $\epsilon=(E-I)/\Gamma$ and substituting in the expressions of § 2, we find that

$$R(\alpha) = \frac{2}{\pi}\int_0^\infty [\exp(-\alpha\epsilon)-1]\frac{d\epsilon}{1+\epsilon^2} = -1 + \frac{2}{\pi}(\sin\alpha \operatorname{Ci}\alpha - \cos\alpha \operatorname{si}\alpha), \quad (6.3)$$

where Ci α, si α are the cosine and sine integrals respectively (cf. Jahnke-Emde, *loc. cit.*, p. 3). For the same reasons as in § 5, we need only consider the asymptotic distribution. It is easily seen from the series expansions of Ci α, si α that to the same order of approximation as in § 4,

$$R(\alpha) = \frac{2}{\pi}\alpha(C-1+\log\alpha), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.4)$$

$$\alpha_p' = \frac{\pi}{4Q}, \quad \text{and hence} \quad \frac{\epsilon_p'}{Q} = \frac{2}{\pi}\,\frac{E_p'-I}{BZt} = \frac{2}{\pi}\left(\log\frac{4Q}{\pi} - C\right). \quad (6.5)$$

Hence we see that as in § 5, if we change over to the reduced energy variable

$$\omega = \frac{\epsilon-\epsilon_p'}{Q} = \frac{2}{\pi}\,\frac{E-E_p'}{BZt} = \log\left(\frac{\pi}{4Q\alpha}\right), \quad . \quad . \quad . \quad . \quad (6.6)$$

then the asymptotic energy distribution takes the 'universal' form

$$\chi_L(\omega)\, d\omega = \frac{1}{\sqrt{(2\pi)}}\exp\{-\tfrac{1}{2}[\omega+\exp(-\omega)]\}\, d\omega. \quad . \quad . \quad (6.7)$$

Thus quantum resonance effects too modify only the expressions for Q and E_p, but not that for $\chi_L(\omega)$, and here again a more exact theory of these effects is unlikely to change these conclusions (though we may expect a modification of the expressions (6.2) and (6.7) for Q and E_p). An estimate of Γ may be obtained by equating the primary ionization rate q with the value calculated by Bethe (1933)

$$q = \frac{BZ}{I_0}\, r\left\{\log\frac{2\mu c^2\beta^2}{I_0(1-\beta^2)} + s - \beta^2\right\} = \frac{2}{\pi}\,\frac{BZ}{\Gamma},$$

where $\beta=v/c$, I_0 is the outer shell ionization potential, r and s are constants depending on the absorber (for hydrogen, $r=0{\cdot}285$ and $s=3{\cdot}04$). With this estimate, Q and E_p exhibit a relativistic log rise with increasing energy similar to the expressions given by Landau (*loc. cit.*). Alternatively, one may estimate Γ from the experimental values found for q.

§ 7. The Distribution of Numbers of Ion Pairs

The production of ions by a fast primary particle constitutes a multiplicative or branching stochastic process: the electron ejected in a primary ionizing collision may, if it has enough energy, ionize another atom; the secondary electrons may then ionize again, and so on, the process ending when all secondaries have become too slow to ionize further. We shall develop the theory under the assumption that the collection or recording of ions is delayed long enough for the process to terminate, so that the ion pair distribution per collision is the same for all collisions: this assumption is valid as a rule in ionization or cloud chamber work: post-expansion cloud chamber tracks form however an exception, since they record only the primary ionization. We shall also neglect extraneous effects such as increase in ionization due to acceleration of the electrons in electrode fields and to photoelectric effects in the gas or chamber walls, which become important in proportional counter work.

Let then q_k be the probability that k ion pairs are produced in any given ionizing collision, with $k=1, 2, \ldots$ and $\Sigma q_k=1$; let $p_n(t)$ be the probability that a total number n of ion pairs is produced in an absorber thickness t, and write q as before for the primary ionizing collision rate. It then follows from assumptions (2.1), (2.2) and (2.3) that

$$p_n(t_1+t_2)=\sum_{j=0}^{n} p_j(t_1)\,p_{n-j}(t_2), \qquad (7.1)$$

$$p_n(t+\delta t)=(1-q\delta t)\delta_{n0}+qq_n\delta t+o(\delta t). \qquad (7.2)$$

Introducing the Laplace transform of the ion pair distribution $p_n(t)$

$$M(\alpha, t)=\sum_{n=0}^{\infty} e^{-n\alpha}p_n(t), \qquad (7.3)$$

we find as in § 2 that

$$M(\alpha, t_1+t_2)=M(\alpha, t_1)\,.\,M(\alpha, t_2), \qquad (7.4)$$

$$\frac{\partial M(\alpha, t)}{\partial t}=\left\{q\sum_{n=1}^{\infty}(e^{-n\alpha}-1)q_n\right\}M(\alpha, t), \qquad (7.5)$$

and hence that

$$M(\alpha, t)=\exp\left\{qt\sum_{n=1}^{\infty}(e^{-n\alpha}-1)q_n\right\}=\exp\,[QR(\alpha)], \qquad (7.6)$$

where $Q=qt$ as before, and

$$R(\alpha)=\sum_{n=1}^{\infty}(e^{-n\alpha}-1)q_n. \qquad (7.7)$$

The inversion of the Laplace transform (7.6) is given by the well-known formula

$$p_n(Q)=\frac{1}{2\pi i}\int_{c-\pi i}^{c+\pi i}\exp\,[QR(z)+nz]\,dz. \quad . \quad . \quad . \quad . \quad (7.8)$$

This integral can be evaluated by the saddle-point method: the first approximation to it (which may be assumed as previously sufficiently accurate for our purpose) is

$$p_n(Q)=\frac{1}{c}\,[2\pi QR''(\alpha_n)]^{-1/2}\exp\,\{Q[R(\alpha_n)-\alpha_nR'(\alpha_n)]\}, \quad . \quad (7.9)$$

where α_n is related to n by

$$n=-QR'(\alpha_n)=Q\sum_{k=1}^{\infty}k\exp\,(-k\alpha_n)\,q_k, \quad . \quad . \quad . \quad (7.10)$$

$$R''(\alpha)=\sum_{k=1}^{\infty}k^2\exp\,(-k\alpha)\,q_k,$$

and c is as before a normalization constant

$$c=\sum_{n=0}^{\infty}[2\pi QR''(\alpha_n)]^{-1/2}\exp\,\{Q[R(\alpha_n)-\alpha_nR'(\alpha_n)]\}$$

$$\simeq\frac{1}{\sqrt{(2\pi)}}\int_{\alpha_0}^{\alpha_\infty}\sqrt{[QR''(\alpha)]}\exp\,\{Q[R(\alpha)-\alpha R'(\alpha)]\}\,d\alpha+\tfrac{1}{2}p_0(t). \quad (7.11)$$

The last approximate expression for c is obtained by the use of Euler's summation formula, with α_0, α_∞ corresponding respectively by (7.10) to $n=0$ and $n=\infty$. These expressions are close analogues of those found in § 2. Similarly, the most probable number of ion pairs is, as in § 2, the number n_p (which need no longer be an integer) corresponding by (7.10) to the solution α_p of

$$R'''(\alpha_p)+2Q\alpha_p[R''(\alpha_p)]^2=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.12)$$

§ 8. Ion Pair Distribution with the Classical Cross Section

We shall now develop the theory of ion pair distribution for the following crude model: we assume the classical cross section for ionization by the secondary electrons as well as by the primary particle, and set this cross section equal to zero if the energy of the electron is less than the mean ionization potential I of the absorber. Let E_0 be the energy of the primary particle, E_1, E_2, ... those of the secondary electrons successively ejected in the chain process resulting from a single primary ionizing collision. The total cross section $\sigma(E\,;\,E_k)\,dE$ for an ionization energy loss between E and $E+dE$ by the kth electron of energy E_k is

$$\sigma(E\,;\,E_k)=\begin{cases}\left(\dfrac{\pi\epsilon^4}{E_k}\right)\dfrac{1}{E^2} & \text{if } E_k\geqslant I \quad\text{and}\quad E\leqslant E_k,\\[2ex] 0 & \text{if } E_k<I \quad\text{or}\quad E>E_k\,;\end{cases} \quad . \quad (8.1)$$

hence, its collision probability per unit thickness is

$$\lambda(E_k)=\begin{cases} N\int_I^{E_k}\sigma(E\,;\,E_k)\,dE=\dfrac{\pi\epsilon^4N}{E_k}\left(\dfrac{1}{I}-\dfrac{1}{E_k}\right) & \text{if } E_k\geq I, \\ 0 & \text{if } E<_kI, \end{cases} \quad (8.2)$$

while the probability that it should *not* ionize further in a thickness t is

$$\kappa(E_k,t)=\exp[-\lambda(E_k)t]=\begin{cases} \exp\left\{-\dfrac{\pi\epsilon^4N}{E_k}\left(\dfrac{1}{I}-\dfrac{1}{E_k}\right)t\right\} & \text{if } E_k\geq I, \\ 1 & \text{if } E_k<I. \end{cases} \quad (8.3)$$

The probability that the primary collision produce a first electron of energy between E_1 and E_1+dE_1 is $IdE_1/(E_1+I)^2$: hence, the probability that just one ion pair be produced in a thickness t is

$$\int_0^{\infty}\kappa(E_1,t)\frac{IdE_1}{(E_1+I)^2}=\tfrac{1}{2}+\frac{I^2}{\pi\epsilon^4Nt}[1-\exp(-\pi\epsilon^4Nt/2I^2)]. \quad . \quad (8.4)$$

The second term in the right-hand side of (8.4) is negligible for any reasonably large value of t (if e.g. $I=25$ ev then for a monoatomic gas $\pi\epsilon^4N/I^2\simeq 2{\cdot}8\times10^3\ \text{cm}^{-1}$); hence the probability q_1 of exactly one ion pair per collision is just the probability that the first electron has an energy $\leq I$: i.e. $q_1=\frac{1}{2}$. Similarly, we can simplify the calculation of $q_2, q_3, \ldots$ by setting $\kappa(E,t)=0$ for $E>I$ and letting $t\to\infty$ in the final result.

The calculations get progressively more complicated for the succeeding q_k; details for q_2 will be found in the Appendix. However, the distribution

$$q_k=\frac{a}{k^2+1}, \qquad (k=1,2,\ldots) \quad . \quad . \quad . \quad . \quad . \quad (8.5)$$

(where a is a normalization constant chosen to make $\Sigma q_k=1$) fits approximately the first few q_k, and has the right sort of behaviour for large k (analogous to the $1/E^2$ law for the distribution of energy loss per collision). The value of a may be obtained from the formula (see e.g. Knopp 1928)

$$\sum_{k=1}^{\infty}\frac{1}{k^2+1}=\frac{\pi}{2}\coth\pi-\tfrac{1}{2}=1{\cdot}077\,; \quad \text{hence} \quad a=0{\cdot}929. \quad . \quad (8.6)$$

The distribution q_k is independent of I: this gives us some confidence in its approximate validity in spite of the neglection of inner shell contributions. In fact, it appears from a rough estimate of the latter's effect, using the model of § 5, that the q_k are not appreciably modified, provided that the inner shell ionization potentials are much greater than that of the outer shell.

Substitution of (8.9) in the expressions of § 7 gives

$$\frac{1}{a}R(\alpha)=\sum_{k=1}^{\infty}\frac{e^{-k\alpha}-1}{k^2+1}\,; \quad \frac{1}{a}R'(\alpha)=-\sum_{k=1}^{\infty}\frac{k\,e^{-k\alpha}}{k^2+1}\,; \quad \frac{1}{a}R''(\alpha)=\sum_{k=1}^{\infty}\frac{k^2\,e^{-k\alpha}}{k^2+1}. \quad . \quad . \quad . \quad . \quad (8.7)$$

The series above converge very slowly for small values of α; for the purposes of computation, we transform the last two as follows:

$$\left.\begin{aligned} \frac{1}{a}R'(\alpha) &= \sum_{k=1}^{\infty}\frac{e^{-k\alpha}}{k(k^2+1)}-\sum_{k=1}^{\infty}\frac{e^{-k\alpha}}{k}=\sum_{k=1}^{\infty}\frac{e^{-k\alpha}}{k(k^2+1)}+\log(1-e^{-\alpha}). \\ \frac{1}{a}R''(\alpha) &= \sum_{k=1}^{\infty}e^{-k\alpha}-\sum_{k=1}^{\infty}\frac{e^{-k\alpha}}{k^2+1}=\frac{1}{e^{\alpha}-1}-\frac{1}{a}[R(\alpha)+1]. \end{aligned}\right\} \quad (8.8)$$

For values of $\alpha<0{\cdot}04$, it is more convenient to compute by numerical integration from the following transformed expression:

$$\left.\begin{aligned} \frac{1}{a}[R(\alpha)+1] &= \int_{\alpha}^{\infty}\frac{\sin(\theta-\alpha)}{e^{\theta}-1}\,d\theta \\ &= -\cos\alpha\left\{\sum_{k=1}^{\infty}\frac{1}{k^2+1}-\int_0^{\alpha}\frac{\sin\theta}{e^{\theta}-1}\,d\theta\right\}+\sin\alpha \\ &\quad\times\left\{\log(1-e^{-\alpha})+\sum_{k=1}^{\infty}\frac{1}{k(k^2+1)}-\int_0^{\alpha}\frac{1-\cos\theta}{e^{\theta}+1}\,d\theta\right\}, \\ \frac{1}{a}R'(\alpha) &= -\int_{\alpha}^{\infty}\frac{\cos(\theta-\alpha)}{e^{\theta}-1}\,d\theta=-\sin\alpha\left\{\sum_{k=1}^{\infty}\frac{1}{k^2+1}-\int_0^{\alpha}\frac{\sin\theta}{e^{\theta}-1}\,d\theta\right\} \\ &\quad+\cos\alpha\left\{\log(1-e^{-\alpha})+\sum_{k=1}^{\infty}\frac{1}{k(k^2+1)}-\int_0^{\alpha}\frac{1-\cos\theta}{e^{\theta}-1}\,d\theta\right\} \end{aligned}\right\} \quad (8.9)$$

The substitution of these expressions in (7.9) and (7.10), taking $Q=BZt/I$, yields the ion pair distribution $p_n(Q)$; in order to compare it with that of the ionization energy, it is important to evaluate its asymptotic expression for large Q. It is easily seen from the transformed expressions (8.9) that to the same order of approximation as in § 4,

$$R(\alpha)=a\alpha(b-1+\log\alpha);\quad \alpha_p'=\frac{1}{2Qa};\quad \frac{n_p'}{Qa}=-b+\log 2Qa; \quad . \quad (8.10)$$

where a is defined by (8.6) and $b=\Sigma 1/k(k^2+1)=0{\cdot}672$. Substituting in (7.9) and changing to the reduced variable

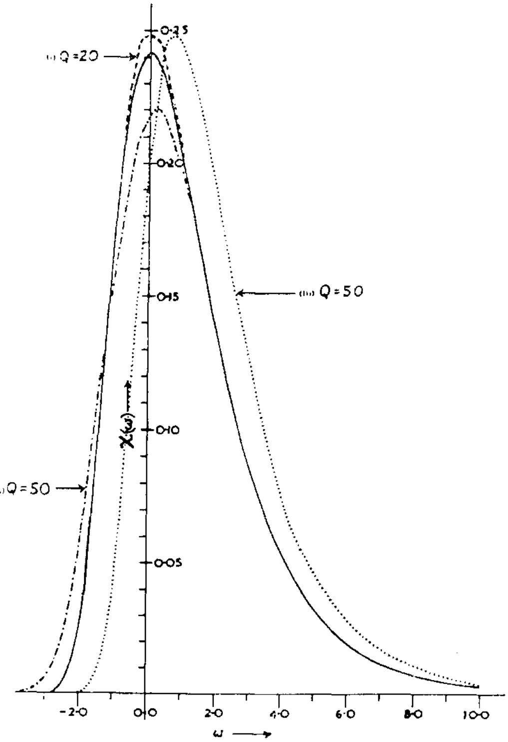
$$\omega=\frac{n-n_p'}{Qa}=-\log 2Qa\alpha, \quad . \quad . \quad . \quad . \quad . \quad (8.11)$$

we are led again, as in the case of ionization energy, to Landau's 'universal' distribution: that is, we find that

$$\left.\begin{aligned} p_n(Q) &= \frac{1}{Qa}\chi_L\left(\frac{n-n_p'}{Qa}\right), \\ \text{where}\quad \chi_L(\omega) &= \frac{1}{\sqrt{(2\pi)}}\exp\{-\tfrac{1}{2}[\omega+\exp(-\omega)]\}. \end{aligned}\right\} \quad . \quad . \quad . \quad (8.12)$$

The deviation from Landau's distribution for small values of Q is somewhat more marked than is the case for the energy loss distribution : it is shown in fig. 4, where $Qap_n(Q)$, computed from expressions (8.8) and (8.9) for $Q=20$, is plotted against ω.

Fig. 4



Distribution of numbers of ion pairs : the full curve is Landau's distribution (8.12), curve (i) is computed with the classical cross section, § 8 for $Q=20$, curves (ii) and (iii) with the quantum resonance cross section, § 9, for $Q=50$ and respectively $q=0{\cdot}17$ and $0{\cdot}50$.

§ 9. Ion Pair Distribution with Quantum Resonance Effects

The effect of quantum resonance is to increase the primary energy loss cross section for values of the energy near the ionization potential I, and hence to increase the probability q_1 of producing a single ion pair per collision. Repeating the calculations of § 8 with the cross section (6.1) for the primary, and the classical cross section for the (slow) ejected electrons, we find that

$$q_1=\frac{2\Gamma}{\pi}\int_0^I\frac{dE_0}{E_0^2+\Gamma^2}=\frac{2}{\pi}\tan^{-1}\frac{I}{\Gamma}. \qquad (9.1)$$

For relativistic primaries, where I/Γ is large, we have approximately

$$q_1\simeq 1-\frac{2}{\pi}\frac{\Gamma}{I}. \qquad (9.1\ a)$$

Continuing the calculations in the same way, one finds that the distribution q_k can be fitted approximately by the relation

$$q_k = g\left(\frac{a}{k^2+1}\right), \quad \text{where} \quad g = \frac{4}{\pi(2-a)}\frac{\Gamma}{I} = 1{\cdot}190\frac{\Gamma}{I}, \quad (k=2, 3, \ldots); \qquad (9.2)$$

one can easily verify that $\sum_1^\infty q_k = 1$. This may be interpreted by saying that at every collision there is a chance $1-g$ of a resonant collision, and a chance g of a non-resonant one, given which the distribution of ion pairs is the same as in § 8.

Let us now write $R_c(\alpha)$, $R_c'(\alpha)$, etc. for the expressions found in § 8, and $R_Q(\alpha)$, $R_Q'(\alpha)$, etc. for those corresponding to the quantum resonance cross section. It then follows from the foregoing that

$$R_Q(\alpha) = (1-g)(e^{-\alpha}-1) + gR_c(\alpha); \quad R_Q'(\alpha) = -(1-g)e^{-\alpha} + gR_c'(\alpha); \qquad (9.3)$$

and so on. To the same order of approximation as in § 4, we then have

$$\left.\begin{aligned} R_Q(\alpha) &= -(1-g)\alpha + ga\alpha(b-1+\log\alpha); \\ R_Q'(\alpha) &= -(1-g) + ga(b+\log\alpha); \quad \text{etc.} \end{aligned}\right\} \quad . \quad . \quad . \quad (9.4)$$

and hence

$$\alpha_p' = \frac{1}{2Qga} = \frac{1}{2.21Q}\frac{I}{\Gamma}; \quad n_p' = Q[1-g+ga(\log 2Qga - b)], \quad . \quad (9.5)$$

where $Q = \pi BZt/2\Gamma$. Substituting in (7.9) in terms of the reduced variable

$$\omega = \frac{n-n_p'}{Qga} = -\log 2Qga\alpha, \quad . \quad . \quad . \quad . \quad . \quad (9.6)$$

we find as in § 8 that

$$\left.\begin{aligned} p_n(Q) &= \frac{1}{Qga}\chi_L\left(\frac{n-n_p'}{Qga}\right), \\ \text{where} \quad \chi_L(\omega) &= \frac{1}{\sqrt{(2\pi)}}\exp\{-\tfrac{1}{2}[\omega+\exp(-\omega)]\}. \end{aligned}\right\} \quad . \quad . \quad . \quad (9.7)$$

The deviation from Landau's distribution is more pronounced than is the case with the ion pair distribution found in § 8, but is still not very marked. This may be seen in fig. 4, where the curves of $Qgap_n(Q)$ as a function of ω, computed from expressions (9.3), are shown for $Q=50$, $g=0{\cdot}5$ and $g=0{\cdot}17$. We may remark that if Γ is evaluated by equating $q=\pi BZ/2\Gamma$ to the experimental values of primary ionization (cf. § 6), one finds that near minimum ionization g is approximately equal to $\frac{1}{2}$ for most gases.

The tendency of the ion pair distribution to the same (Landau) distribution as the primary energy loss accounts (at least in the case of fast primaries) for the experimental fact that the energy lost per ion pair is a constant, approximately independent of the primary energy: furthermore, we may presume that this result is not sensitive to the actual value of the cross section, since it holds for both the classical

(§ 8) and the quantum resonance (§ 9) cross sections. Comparing the expressions for ω in §§ 8 and 9 with those respectively in §§ 4 and 6, we see that the energy per ion pair must be approximately equal to the average ionization potential. This is substantially less than the experimental values (of the order of 35 ev), presumably because the latter refers to total energy loss, including excitation of the atoms, whereas our calculations refer to energy lost by ionization only. Furthermore, the use of a single average ionization potential is undoubtedly a very rough approximation, especially at the higher atomic numbers, where inner shell ionization and excitation, Auger effects and the like must contribute substantially to ionization and primary energy loss. We cannot in fact expect the theory in the present section to have more than qualitative validity for elements heavier than helium.

§ 10. Conclusions

We conclude from the foregoing:

(1) That Landau's 'universal' distribution appears to be valid for both primary energy loss and numbers of ion pairs, which explains their proportionality in the case of fast primaries.

(2) That it remains valid down to unexpectedly small values of the primary ionization Q.

(3) That none of the effects considered in this paper (atomic structure, quantum resonance, proportionality of numbers of ion pairs to primary energy loss) explain the experimentally found deviations from it.

Acknowledgments

It is a pleasure for me to acknowledge my indebtedness to Dr. B. G. Owen and Mr. D. A. Eyeions, at whose request I considered the present problem, for many invaluable discussions and suggestions. I am also greatly indebted to Mrs. A. Linnert, who carried out the computations and drew the graphs.

References

Bethe, 1933, *Handbuch der Physik* (Berlin: Springer), Vol. **24,** 1, p. 518.

Blunck and Leisegang, 1950, *Z. Phys.*, **128,** 500.

Blunck and Westphal, 1951, *Z. Phys.*, **130,** 641.

Bohr, 1948, *The Penetration of Atomic Particles through Matter* (Copenhagen: Det Kgl. Danske Videnkabernes Selskab), pp. 64, 76, 80.

British Association Tables, 1951 (Cambridge: University Press), Vol. **1,** 3rd edn., pp. 31–33.

Daniels, 1955, *Annals of Math. Statistics* (to be published).

Jahnke-Emde, 1938, *Tables of Functions* (New York: Dover Publications), 2nd edn., pp. 6–8.

Jeffreys and Jeffreys, 1946, *Methods of Mathematical Physics* (Cambridge: University Press), 1st edn., pp. 472–6.

Knopp, 1928, *Theory and Application of Infinite Series* (London and Glasgow: Blackie), p. 378.

Landau, 1944, *J. Phys. U.S.S.R.*, **8,** 201.

West, 1953, *Proc. Phys. Soc.* A, **66,** 306.

APPENDIX

Calculation of q_2

In order to evaluate q_2, we note first that the probability that electron 1, of energy $E_1 \geqslant I$, suffer an ionizing collision between thicknesses τ and $\tau+d\tau$, thereby producing electrons 2 and 3 with energies respectively between E_2, E_2+dE_2 and E_3, E_3+dE_3, where $E_3=E_1-E_2-I$, and that electrons 2 and 3 ionize no further between thicknesses τ and t, is

$$\kappa(E_1\tau) \, . \, N\sigma(E_2\,;\,E_1)\, dE_2\, d\tau \, . \, \kappa(E_2, t-\tau) \, . \, \kappa(E_1-E_2-I, t-\tau), \qquad \text{(A1)}$$

where σ and κ are given respectively by eqns. (8.1) and (8.3). We then obtain q_2 by multiplying (A1) by the probability distribution $IdE_1/(E_1+I)^2$ of electron 1, integrating with respect to E_2 (from 0 to E_1-I), to E_1 (from I to ∞), to τ (from 0 to t), and finally making $t\rightarrow\infty$: thus

$$q_2=\lim_{t\rightarrow\infty}\int_I^\infty \frac{IdE_1}{(E_1+I)^2}\int_0^t \kappa(E_1, \tau)\, d\tau \int_0^{E_1-I} \sigma(E_2\,;\,E_1)\kappa(E_2, t-\tau)$$
$$\times\,\kappa(E_1-E_2-I, t-\tau)\, dE_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(A2)}$$

The only positive contribution to q_2 in the above comes from the range of values of E_1 and E_2 which make E_2 and $E_3=E_1-E_2-I$ both $<I$, and hence $I \leqslant E_1 < 3I$. If $E_1 \geqslant 3I$, then one at least of electrons 2 and 3 will have an energy $\geqslant I$, and will therefore ionize further. If $I \leqslant E_1 < 2I$, then $E_2+E_3<I$, and therefore both E_1 and E_2 are $<I$, and will not ionize; hence, we get from this range the contribution

$$\int_I^{2I} \frac{IdE_1}{(E_1+I)^2}=\frac{1}{6}.$$

If $2I \leqslant E_1 < 3I$, then the requirement that $E_2<I$ and $E_1-E_2-I<I$ means that the range of E_2 must be limited to $E-2I<E_2<I$. The last two integrals in the right-hand side of (A2) yield in this range

$$\int_0^\infty \exp\left[-\lambda(E_1)\tau\right] d\tau \int_{E_1-2I}^{I} \frac{\pi\epsilon^4 N}{E_1}\, \frac{dE_2}{(E_2+I)^2}=\frac{E_1(3I-E_1)}{2(E_1-I)^2}\,;$$

integrating with respect to E_1, we obtain the contribution

$$\int_{2I}^{3I} \frac{E_1(3I-E_1)}{2(E_1-I)^2}\cdot\frac{IdE_1}{(E_1+I)^2}=\frac{1}{12}-\frac{1}{8}\log\frac{3}{2}.$$

Adding these two contributions, we finally obtain

$$q_2=\frac{1}{4}-\frac{1}{8}\log\frac{3}{2}=0{\cdot}199. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(A3)}$$

Note added in proof.—Contrary to the conclusions stated above, it now appears from a critical analysis by Dr. E. P. George of the available experimental data that agreement with the theory is substantially improved, at least for gases, by the results of § 6 (owing to the broadening of the scale for the reduced distribution by a factor $\pi/2$: cf. eqn. (6.6)); this was also pointed out to me in a letter by Dr. B. T. Price. It now seems possible that, with the introduction of a more exact cross section, the theory should account for all the facts.

Dr. U. Fano has pointed out that the approximate distribution (4.6) does not agree with Landau's numerical evaluation for large ω owing to the breakdown of the saddle-point method in this range. This is not very important as regards comparison with experiment, because the frequency of events with energy loss greater than twice the probable value is small (of the order of 5%).