THE BETA MOYAL: A USEFUL-SKEW DISTRIBUTION

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ABSTRACT

For the first time, we propose the called beta Moyal distribution that generalizes the Moyal distribution, and study its properties. We derive expansions for the cumulative distribution function as power series of the Moyal cumulative distribution. We derive expansions for its moments, generating function, mean deviations, density function of the order statistics and their moments. We discuss maximum likelihood estimation of the model parameters. We illustrate the superiority of the new distribution as compared to the beta normal, skew-normal and Moyal distributions by means of three real data sets.

Keywords: *Entropy; Expected information; Maximum likelihood estimation; Moment; Moyal distribution; Order Statistic.*

1. INTRODUCTION

One **m**ajor benefit of the class of beta generalized distributions proposed by Eugene *et al.* (2002) is its ability of fitting skewed data that can not be properly fitted by existing distributions. Starting from a parent cumulative distribution function (cdf) $G(x)$, this class is defined by

$$F(x) = I_{G(x)}(a,b) = \frac{1}{B(a,b)}\int_0^{G(x)} \omega^{a-1}(1-\omega)^{b-1}d\omega, \tag{1}$$

where a and b are additional positive parameters, $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ is the beta function, $\Gamma(a)$ is the gamma function, $I_y(a,b) = B_y(a,b)/B(a,b)$ is the incomplete beta function ratio and $B_y(a,b) = \int_0^y w^{a-1}(1-w)^{b-1}dw$ is the incomplete beta function. This class of generalized distributions has been receiving considerable attention over the last years in particular after the work of Jones (2004).

Eugene *et al.* (2002), Nadarajah and Kotz (2004), Nadarajah and Gupta (2004), Nadarajah and Kotz (2005), Lee *et al.* (2007) and Akinsete *et al.* (2008) defined the beta normal, beta Gumbel, beta Fréchet, beta exponential, beta Weibull and beta Pareto distributions by taking $G(x)$ to be the cdf of the normal, Gumbel, Fréchet, exponential, Weibull and Pareto distributions, respectively. More recently, Pescim *et al.* (2010) and Barreto-Souza *et al.* (2010) studied the beta generalized half-normal and beta generalized exponential distributions, respectively.

The probability density function (pdf) corresponding to (1) is

$$f(x) = \frac{g(x)}{B(a,b)}G(x)^{a-1}\{1-G(x)\}^{b-1}, \tag{2}$$

where $g(x) = dG(x)/dx$ is the parent density function. The density $f(x)$ will be most tractable when both functions $G(x)$ and $g(x)$ have simple analytic expressions. Except for some special choices of these functions, $f(x)$ will be difficult to deal with some generality.

In this note, we introduce a four parameter model, called the beta Moyal (BMo) distribution, to extend the Moyal distribution. The BMo distribution is convenient for modeling comfortable upside-down bathtub-shaped failure rates and as a competitive model to the Moyal, half-normal, beta normal, skew normal and Gumbel distributions.

The article is organized as follows. In Section 2, we define the BMo distribution, present some special sub-models and provide expansions for its distribution and density functions. Section 3 gives general expansions for the moments, moment generating function (mgf), mean deviations and Rényi entropy. In Section 4, we derive expansions for the moments of order statistics. Maximum likelihood estimation and inference issues are addressed in Section 5. Section 6 illustrates the importance of the BMo distribution by means of three real data sets. Finally, concluding remarks are given in Section 7.

2. BETA MOYAL DISTRIBUTION

The Moyal distribution (Moyal, 1955) was proposed as an approximation to the Landau distribution. It was also shown that it remains valid taking into account quantum resonance effects and details of atomic structure of the absorber. The Moyal distribution is a universal form for the energy loss by ionization for a fast charged particle and the number of ion pairs produced in this process. Let X be a random variable having the Moyal standard density function given by

$$g_X(x) = \frac{1}{\sqrt{2\pi}} \exp\{-[x+\exp(-x)]/2\}, \quad -\infty < x < \infty. \tag{3}$$

A location parameter μ and a scale factor σ can be introduced to define the random variable $Z = \sigma X + \mu$ having a Moyal distribution, say Mo (μ, σ), given by

$$g_Z(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2}\left[(\frac{x-\mu}{\sigma}) + \exp\left\{-(\frac{x-\mu}{\sigma})\right\}\right]\right\}, \tag{4}$$

where $-\infty < x, \mu < \infty$ and $\sigma > 0$. The cumulative function corresponding to (4) depends on the incomplete gamma function $\gamma(\alpha, x) = \int_0^x t^{\alpha-1} \mathrm{e}^{-t} dt$. It is given by

$$G_Z(x) = 1 - \frac{\gamma\left\{\frac{1}{2}, \frac{1}{2}\exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right]\right\}}{\Gamma(\frac{1}{2})}. \tag{5}$$

The cumulants of the standard Moyal distribution (3) are

$$\kappa_1 = -\log(2) - \psi(\frac{1}{2}) = \log(2) + \gamma \text{ and } \kappa_n = (-1)^n \psi^{n-1}(\frac{1}{2}) = (n-1)!(2^n - 1)\zeta(n), n \geq 2,$$

where $\gamma \approx 0.57721$ is Euler's constant, $\psi^{(n)}$ denotes polygamma functions and $\zeta(\cdot)$ is the Riemann's zeta function defined by

$$\zeta(u) = \sum_{k=1}^{\infty} \frac{1}{k^u} = \frac{1}{\Gamma(u)} \int_0^{\infty} \frac{x^{u-1}}{\mathrm{e}^x - 1} dx \quad \textit{for} \quad u > 1.$$

The moments can be easily obtained from these cumulants. Those of lower order are $\mu_1' = E(X) = \log(2) + \gamma \approx 1.27036$, $\mu_2' = Var(X) = \pi^2/2 \approx 4.9348$, $\mu_3' = 14\zeta_3$ and $\mu_4' = 7\pi^4/4$. For the distribution (4), $E(Z) = \sigma E(X) + \mu$, $Var(Z) = \sigma^2 Var(X)$ and, more generally, the central moments of Z ($\mu_{n,Z}$) are easily obtained from the central moments of X ($\mu_{n,X}$) by $\mu_{n,Z} = \sigma^n \mu_{n,X}$ for $n \geq 2$.

The characteristic function of (3) is

$$\phi_X(t) = E(\mathrm{e}^{itX}) = \frac{2^{-\mathrm{i}t}}{\sqrt{\pi}} \Gamma(\frac{1}{2} - \mathrm{i}t),$$

where $\mathrm{i} = \sqrt{-1}$. The gamma function $\Gamma(\cdot)$ with complex argument is defined when the real part of the argument is positive, which is indeed true in this case.

The Moyal distribution can be defined in a finite interval. In fact, the transformation $X = \tan(Y)$ gives the density function of Y as

$$\pi(y) = \frac{1}{\sqrt{2\pi}\cos^2(y)} \exp\{-\frac{1}{2}[\tan(y) + \exp\{-\tan(y)\}]\}, \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}.$$

This density function has a maximum of about 0.91 and it is widely used to generate Moyal variates.

Now, we introduce the four parameter BMo distribution by taking $G(x)$ in (1) to be the cdf (5) of the Moyal distribution. The BMo cumulative function is given by

$$F(x) = \frac{1}{B(a,b)} \int_0^{1-\frac{\gamma\left\{\frac{1}{2},\frac{1}{2}\exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right]\right\}}{\Gamma(\frac{1}{2})}} \omega^{a-1}(1-\omega)^{b-1} d\omega. \tag{6}$$

Inserting (4) and (5) into (2) gives the BMo density function

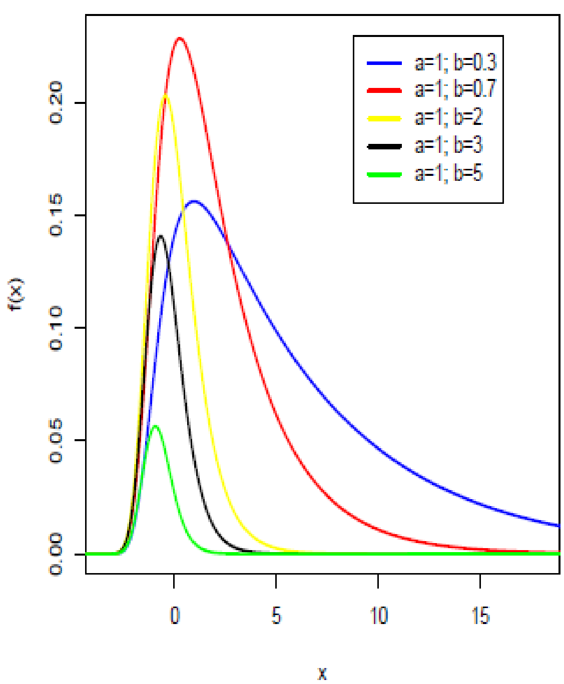
$$f(x) = \frac{\exp\{-\frac{1}{2}[(\frac{x-\mu}{\sigma}) + \exp\left\{-(\frac{x-\mu}{\sigma})\right\}]\}}{\sqrt{2\pi}\,\sigma\Gamma\left(\frac{1}{2}\right)^{b-1} B(a,b)} \left\{1 - \frac{\gamma\left\{\frac{1}{2},\frac{1}{2}\exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right]\right\}}{\Gamma(\frac{1}{2})}\right\}^{a-1} \times$$

$$\left\{\gamma\left(\frac{1}{2},\frac{1}{2}\exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right]\right)\right\}^{b-1}, \quad -\infty < x < \infty, \tag{7}$$

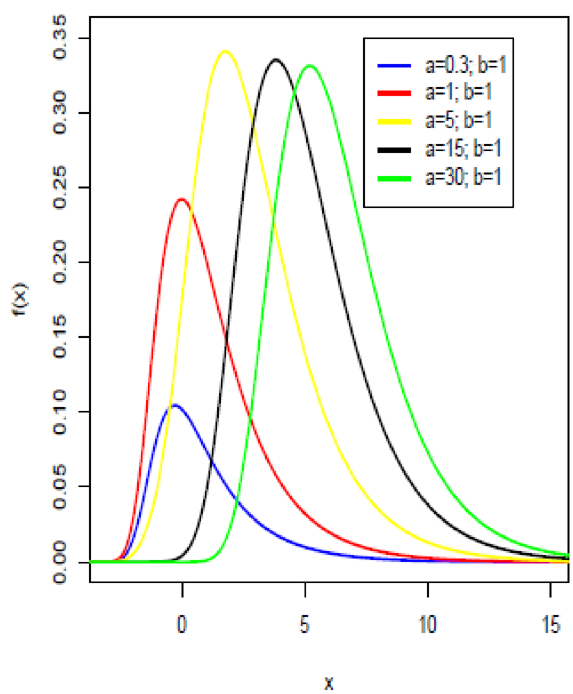
where $-\infty < \mu < \infty$ is the location parameter, $\sigma > 0$ is the scale parameter and $a > 0$ and $b > 0$ are shape parameters. For $a = b = 1$, it reduces to the Moyal distribution. For $\mu = 0$ and $\sigma = 1$, we obtain the standard BMo density function given by

$$f(x) = \frac{\exp\{-\frac{1}{2}[x + \exp(-x)]\}}{\sqrt{2\pi}\,\Gamma\left(\frac{1}{2}\right)^{b-1} B(a,b)} \left\{1 - \frac{\gamma\left[\frac{1}{2},\frac{\exp(-x)}{2}\right]}{\Gamma\left(\frac{1}{2}\right)}\right\}^{a-1} \left\{\gamma\left[\frac{1}{2},\frac{\exp(-x)}{2}\right]\right\}^{b-1}. \tag{8}$$

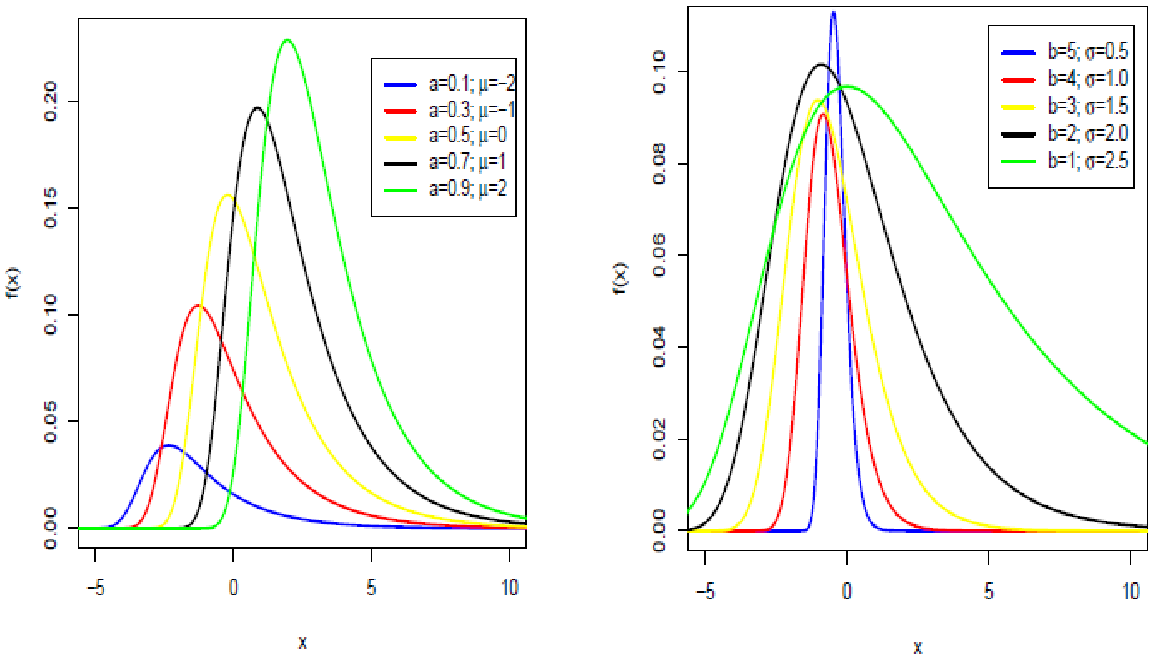
Plots of the density function (7) for selected parameter values are given in Figure 1. These plots show great flexibility of the new distribution for different values of the shape parameters a and b, including the special case of the standard Moyal distribution. The density function (7) allows for great flexibility and then it can be very useful in many more practical situations, i.e. the BMo distribution can be symmetric and asymmetric.



(a)



(b)



(c) (d)

Figure 1: Plots of the density function (7) for some parameter values. (a) $\mu = 0$ and $\sigma = 1$. (b) $\mu = 0$ and $\sigma = 1$. (c) $b = 1$ and $\sigma = 1$. (d) $\mu = 0$ and $a = 1$.

If X is a random variable with density function (7), we write $X \sim \mathrm{BMo}(a,b,\mu,\sigma)$. The BMo distribution is easily simulated from $F(x)$ in (6) as follows: if V has a beta distribution with parameters a and b, then the solution of the nonlinear equation

$$\frac{X-\mu}{\sigma} = -\log\left\{2\left[erf^{-1}(1-V)\right]^2\right\}$$

gives the BMo (a,b,μ,σ) distribution, where $\gamma\left(\frac{1}{2},x\right) = \sqrt{\pi}\, erf(\sqrt{x})$ and $erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$ is the error function. To simulate data from this nonlinear equation, we can use the programming language Ox through the SolveNLE subroutine (see Doornik, 2007).

We provide two simple formulae for the cdf of the BMo distribution depending if the parameter $b > 0$ is real non-integer or integer. First, if $|z| < 1$ and $b > 0$ is real non-integer, we have the series representation

$$(1-z)^{b-1} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{j!\Gamma(b-j)} z^j. \tag{9}$$

For $b > 0$ real non-integer, by the representation (9), the standard cumulative function (6) (for $\mu = 0$ and $\sigma = 1$) can be expanded as

$$F(x) = \frac{\Gamma(a+b)}{\Gamma(a)} \sum_{j=0}^{\infty} \frac{(-1)^j \left\{1 - \frac{\gamma\left(\frac{1}{2}, \frac{e^{-x}}{2}\right)}{\Gamma(\frac{1}{2})}\right\}^{a+j}}{j!\,(a+j)\,\Gamma(b-j)}. \tag{10}$$

If $a > 0$ is an integer, (10) gives the cdf of the BMo distribution in terms of a power series of the Moyal cumulative function. Otherwise, if $a > 0$ is real non-integer, inserting expansion (9) in (10) gives

$$F(x) = \frac{\Gamma(a+b)}{\Gamma(a)} \sum_{j,r=0}^{\infty} \frac{(-1)^{j+r}\, \gamma\left(\frac{1}{2}, \frac{\exp(-x)}{2}\right)^{r}}{j!\, r!\, \Gamma(b-j)\, \Gamma(a+j+1-r)\, \Gamma(\frac{1}{2})^{r}}. \tag{11}$$

For both a and b real non-integers, equation (11) reveals that the BMo cumulative distribution can be expressed as an infinite power series of the incomplete gamma function.

By application of the binomial expansion in (6), when $b > 0$ is an integer, we obtain

$$F(x) = \frac{1}{B(a,b)} \sum_{j=0}^{b-1} \binom{b-1}{j} \frac{(-1)^{j}}{(a+j)} \left\{ 1 - \frac{\gamma\left(\frac{1}{2}, \frac{e^{-x}}{2}\right)}{\Gamma(\frac{1}{2})} \right\}^{a+j}. \tag{12}$$

For $a > 0$ integer, applying the binomial expansion in (12), yields

$$F(x) = \frac{1}{B(a,b)} \sum_{j=0}^{b-1} \sum_{r=0}^{a+j} \frac{(-1)^{j+r}}{(a+j)\, \Gamma(\frac{1}{2})^{r}} \binom{b-1}{j} \binom{a+j}{r} \gamma\left(\frac{1}{2}, \frac{e^{-x}}{2}\right)^{r}. \tag{13}$$

For $a > 0$ real non-integer, expanding (12) as in (9), we have

$$F(x) = \frac{1}{B(a,b)} \sum_{j=0}^{b-1} \sum_{r=0}^{\infty} \frac{(-1)^{j+r}\, 2^{r}\, \Gamma(a+j+1)}{(a+j)\, r!\, \Gamma(a+j+1-r)\, \Gamma(\frac{1}{2})^{r}} \binom{b-1}{j} \binom{r}{s} \gamma\left(\frac{1}{2}, \frac{e^{-x}}{2}\right)^{r}. \tag{14}$$

The standard Moyal cumulative function can be obtained from equation (12) when $a = b = 1$. Equations (10)-(14) are the main expansions for the cdf of the BMo distribution. They (and other expansions in the paper) can be evaluated in symbolic computation software such as Mathematica and Maple}.These symbolic software have currently the ability to deal with analytic expressions of formidable size and complexity.

Alternatively to (8), an expansion for the standard BMo density function for b real non-integer follows by differentiating (10) and using the series representation (9)

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\{-[x + e^{-x}]/2\} \sum_{k=0}^{\infty} w_k(a,b) \left\{ \gamma\left(\frac{1}{2}, \frac{e^{-x}}{2}\right) \right\}^{k}, \tag{15}$$

whose coefficients $w_k(a,b)$ are

$$w_k(a,b) = \sum_{j=0}^{\infty} \frac{(-1)^{j+k}\, \Gamma(b)\, \Gamma(a+j)}{k!\, j!\, \Gamma(b-j)\, \Gamma(a+j-k)\, \Gamma(\frac{1}{2})^{k}\, B(a,b)}.$$

Equation (15) is the basic expansion for the standard BMo density function.

3. PROPERTIES

We hardly need to emphasize the necessity and importance of moments and generating function in any statistical analysis especially in applied work. Some of the most important features and characteristics of a distribution can be studied through moments (e.g., tendency, dispersion, skewness and kurtosis).

3.1 Moments

Theorem 1: If $X \sim BMo(a,b,0,1)$, the s th moment of X is given by

$$\mu_s^{'} = \frac{1}{\sqrt{\pi}} \sum_{k,m=0}^{\infty} \sum_{r=0}^{s} v_{k,r,s,m}(a,b)\, \Gamma_r\left(m + \frac{k+1}{2}\right), \tag{16}$$

where all quantities are defined in the following proof.

Proof:

The s th moment of the BMo distribution is $\mu_s^{'} = \int_{-\infty}^{\infty} x^s f(x)\, dx$. Hence, if $b > 0$ is real non-integer, we obtain from (15)

$$\mu_s^{'} = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} w_k(a,b) \int_{-\infty}^{\infty} x^s \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}[x + \exp(-x)]\} \left\{\gamma\left[\frac{1}{2}, \frac{\exp(-x)}{2}\right]\right\}^k dx.$$

Setting $u = \exp(-x)/2$, $\mu_s^{'}$ reduces to

$$\mu_s^{'} = \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} w_k(a,b)\, (-1)^s \int_0^{\infty} u^{-\frac{1}{2}} \log^s(2u)\; \mathrm{e}^{-u} \left[\gamma\left(\frac{1}{2}, u\right)\right]^k du.$$

Using the binomial expansion in the last equation, we can obtain

$$\mu_s^{'} = \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \sum_{r=0}^{s} w_k(a,b)\, (-1)^s \binom{s}{r} \log^{(s-r)}(2) \int_0^{\infty} u^{-\frac{1}{2}} \log^r(u)\; \mathrm{e}^{-u} \left[\gamma\left(\frac{1}{2}, u\right)\right]^k du. \tag{17}$$

By the series expansion $\gamma(\alpha, x) = x^{\alpha} \sum_{m=0}^{\infty} \frac{(-x)^m}{(\alpha + m)\, m!}$, we can rewrite the integral in (17), say $I(r,k)$, using the identity of a power series raised to an integer, namely $(\sum_{k=0}^{\infty} a_k x^k)^n = \sum_{k=0}^{\infty} c_{k,n} x^k$ (see Gradshteyn and Ryzhik, 2000), where $c_{0,n} = a_0^n$ and $c_{k,n} = (k a_0)^{-1} \sum_{l=1}^{k} (nl - k + l) a_l c_{k-l,n}$.
Hence,

$$I(r,k) = \int_0^{\infty} u^{-\frac{1}{2}} \log^r(u)\; \mathrm{e}^{-u} \left[u^{\frac{1}{2}} \sum_{m=0}^{\infty} \frac{(-u)^m}{(\frac{1}{2} + m)\, m!}\right]^k du = \int_0^{\infty} u^{\frac{k-1}{2}} \log^r(u)\; \mathrm{e}^{-u} \sum_{m=0}^{\infty} c_{m,k} u^m\, du,$$

where $c_{m,k} = m^{-1} \sum_{l=1}^{m} \frac{(-1)^l (kl - m + l)}{(2l+1)\, l!} c_{m-l,k}$ for $m = 1,2,\ldots$ and $c_{0,k} = 2^k$, $k = 1,2,\ldots$ Inserting the last equation in (17) yields

$$\mu_s^{'} = \frac{1}{\sqrt{\pi}} \sum_{k,m=0}^{\infty} \sum_{r=0}^{s} w_k(a,b)\; c_{m,k}\; (-1)^s \binom{s}{r} \log^{(s-r)}(2) \int_0^{\infty} u^{m + \frac{k+1}{2} - 1} \log^r(u)\; \mathrm{e}^{-u}\, du. \tag{18}$$

The integral $J(r)$ in (18) can be easily determined from a result given by Prudnikov *et al.* (1986, Vol.1, Section 2.6.21, integral 1). From the definition of $\Gamma_r(p) = \frac{\partial^r \Gamma(p)}{\partial p^r}$, we have

$$J(r) = \int_0^{\infty} u^{m + \frac{k+1}{2} - 1} \log^r(u)\; \mathrm{e}^{-u}\, du = \Gamma_r\left(m + \frac{k+1}{2}\right).$$

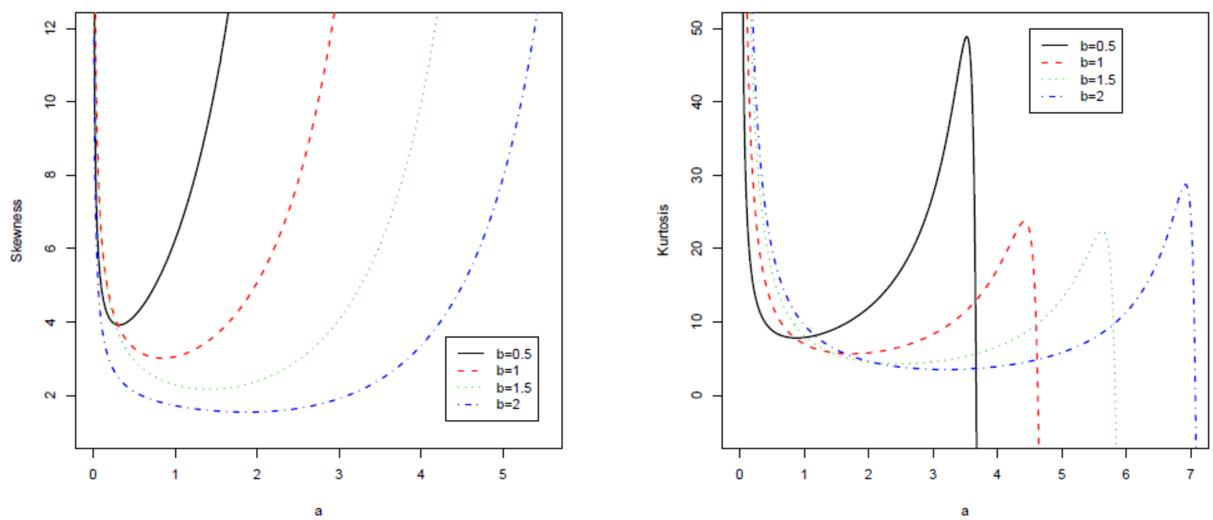
Hence, the s th moment of the standard BMo distribution can be expressed as

$$\mu_s^{'} = \frac{1}{\sqrt{\pi}} \sum_{k,m=0}^{\infty} \sum_{r=0}^{s} v_{k,r,s,m}(a,b)\, \Gamma_r\left(m + \frac{k+1}{2}\right),$$

where

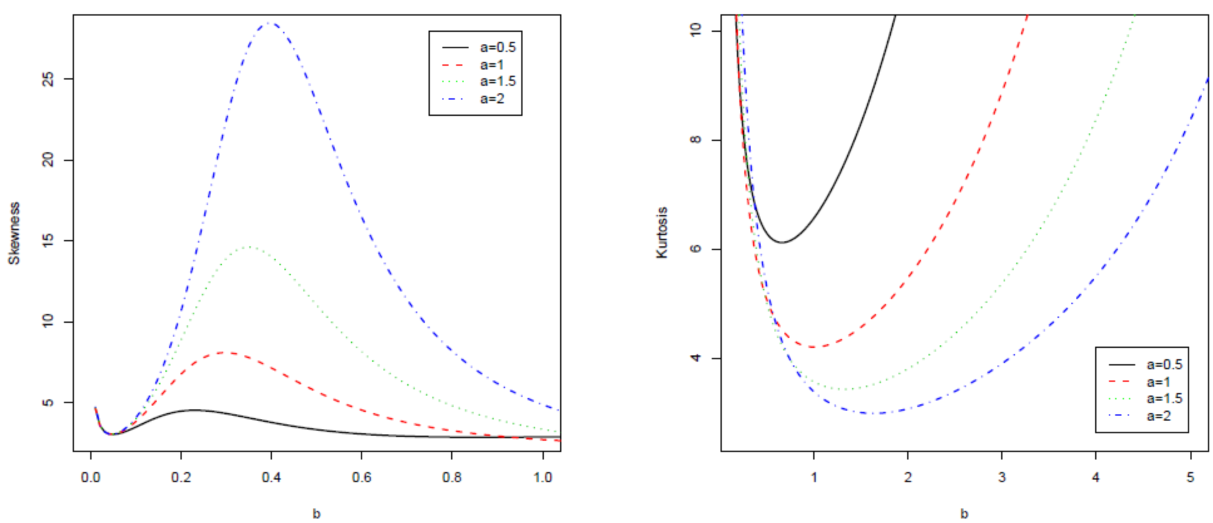
$$v_{k,r,s,m}(a,b) = w_k(a,b)\; c_{m,k}\; (-1)^s \binom{s}{r} \log^{(s-r)}(2). \quad +$$

The skewness and kurtosis measures can be determined calculated from the ordinary moments using well-known relationships. Plots of the skewness and kurtosis for some choices of the parameter b as function of a, and for some choices of the parameter a as function of b, for $\mu = 0$ and $\sigma = 1$, are shown in Figures 1 and 2, respectively.



(a) (b)

Figure 2: Skewness and kurtosis of the standard BMo distribution as a function of a for selected values of b.



(a) (b)

Figure 3: Skewness and kurtosis of the standard BMo distribution as a function of b for selected values of a.

3.2 Generating Function

Theorem 2: If $X \sim BMo(a,b,0,1)$, the mgf of X reduces to

$$M(t) = \frac{2^{-t}}{\sqrt{\pi}} \sum_{k,m=0}^{\infty} w_k(a,b)\; c_{m,k}\; \Gamma\left(m + \frac{k+1}{2} - t\right),$$

where $w_k(a,b)$ and $c_{m,k}$ are defined in Sections 2 and 3.1, respectively.

Proof:

The mgf of the standard BMo distribution is

$$M(t) = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} w_k(a,b) \int_{-\infty}^{\infty} \exp(tx) \exp\left\{-\frac{1}{2}\left[x + \exp(-x)\right]\right\} \left\{\gamma\left[\frac{1}{2}, \frac{\exp(-x)}{2}\right]\right\}^k dx.$$

Substituting $u = \mathrm{e}^{-x}/2$, we have

$$M(t) = \frac{2^{-t}}{\sqrt{\pi}} \sum_{k=0}^{\infty} w_k(a,b) \int_0^{\infty} u^{-\frac{1}{2}-t}\; \mathrm{e}^{-u} \left[\gamma\left(\frac{1}{2}, u\right)\right]^k du. \tag{19}$$

Following similar steps of Theorem 1, $M(t)$ takes the form

$$M(t) = \frac{2^{-t}}{\sqrt{\pi}} \sum_{k,m=0}^{\infty} w_k(a,b)\; c_{m,k} \int_0^{\infty} u^{m+\frac{k+1}{2}-t-1}\; \mathrm{e}^{-u} du.$$

By the definition of the gamma function, we obtain the stated result.

3.3 Means Deviations

The amount of scatter in a population is evidently measured to some extent by the totality of deviations from the mean and median. The mean deviations of X about the mean and the median are defined by

$$\delta_1(X) = \int_{-\infty}^{\infty} |x - \mu_1'|\, dx \quad and \quad \delta_2(X) = \int_{-\infty}^{\infty} |x - M|\, dx,$$

respectively, where $\mu_1' = E(X)$ and $M = Median(X)$ is the median. If $X \sim BMo(a,b,0,1)$, these measures can be expressed as

$$\delta_1(X) = 2\mu_1' F(\mu_1') - 2T(\mu_1') \text{ and } \delta_2(X) = \mu_1' - 2T(M),$$

where $T(q) = \int_{-\infty}^{q} x f(x) dx$, the median M satisfies the nonlinear equation

$$I_{\left[1 - \frac{\gamma\left(1/2, \mathrm{e}^{-x}/2\right)}{\Gamma(1/2)}\right]}(a,b) = 1/2,$$

and $F(\mu_1')$ and $F(M)$ are given by (6). From (15), we have

$$T(q) = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} w_k(a,b) \int_{-\infty}^{q} x\; \exp\{-[x + \exp(-x)]/2\} \left\{\gamma\left[\frac{1}{2}, \frac{\exp(-x)}{2}\right]\right\}^k dx.$$

The transformation $u = \mathrm{e}^{-x}/2$ leads to

$$T(q) = -\frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} w_k(a,b) \int_{\frac{1}{2}\exp(-q)}^{\infty} u^{-\frac{1}{2}} \log(2u) \exp(-u) \left[\gamma\left(\frac{1}{2}, u\right)\right]^k du.$$

Following similar steps from Theorem 1, we can rewrite $T(q)$ as

$$T(q) = -\frac{1}{\sqrt{\pi}} \sum_{k,m=0}^{\infty} w_k(a,b)\; c_{m,k} \left\{\log(2)\, \Gamma\left(m + \frac{k+1}{2}, \frac{1}{2}\exp(-q)\right)\right.$$

$$+\int_{\frac{1}{2}\exp(-q)}^{\infty} u^{m+\frac{k+1}{2}-1} \log(u)\exp(-u)\,du \Bigg\},$$

where $\Gamma\left(m+\frac{k+1}{2},\frac{\exp(-q)}{2}\right)=\int_{\frac{\exp(-q)}{2}}^{\infty} u^{m+\frac{k+1}{2}-1}\exp(-u)\,du$ is the complementary gamma function and $c_{m,k}$ is defined in Section 3.1. Then,

$$T(q)=-\frac{1}{\sqrt{\pi}}\sum_{k,m=0}^{\infty} w_k(a,b)\,c_{m,k}\Bigg\{\log(2)\,\Gamma\left(m+\frac{k+1}{2},\frac{\exp(-q)}{2}\right)$$
$$+\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\int_{\frac{\exp(-q)}{2}}^{\infty} u^{m+r+\frac{k+1}{2}-1}\log(u)\,du\Bigg\}.$$

Calculating the integral in the last equation by Maple, we obtain

$$T(q)=-\frac{1}{\sqrt{\pi}}\sum_{k,m=0}^{\infty} w_k(a,b)\,c_{m,k}\Bigg\{\log(2)\,\Gamma\left(m+\frac{k+1}{2},\frac{1}{2}\exp(-q)\right)$$
$$+\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left[\left(m+r+\frac{k+1}{2}\right)\frac{q}{2}+1\right]\left(m+r+\frac{k+1}{2}\right)^{-2}\left[\frac{\exp(-q)}{2}\right]^{\left(m+r+\frac{k+1}{2}\right)}\Bigg\}. \quad (20)$$

The measures $\delta_1(X)$ and $\delta_2(X)$ are immediately determined from (20).

An application of the mean deviations is to obtain the Lorenz and Bonferroni curves, which are important in several fields such as economics, reliability, demography, insurance and medicine. For a given probability π, they are defined by $L(\pi)=T(q)/\mu_1'$ and $B(\pi)=T(q)/(\pi\mu_1')$, respectively, where $q=Q(\pi)=F^{-1}(\pi)$ is determined from the beta quantile function with parameters a and b (say $Q_{a,b}(\pi)$) by $q=-\log\left\{2\left[erf^{-1}(1-Q_{a,b}(\pi))\right]^2\right\}$.

In economics, if $\pi=F(q)$ is the proportion of units whose income is lower than or equal to q, $L(\pi)$ gives the proportion of total income volume accumulated by the set of units with an income lower than or equal to q. The Lorenz curve is increasing and convex and given the mean income, the density function of X can be obtained from the curvature of $L(\pi)$. In a similar manner, the Bonferroni curve $B(\pi)$ gives the ratio between the mean income of this group and the mean income of the population. In summary, $L(\pi)$ yields fractions of the total income, while the values of $B(\pi)$ refer to relative income levels.

3.4 Rényi Entropy

The entropy of a random variable is a measure of variation of the uncertainty. Entropy has been used in various situations in science and engineering, and numerous measures of entropy have been studied and compared in literature. The Rényi entropy is defined by

$$j_R(\xi)=\frac{1}{1-\xi}\log\left[I(\xi)\right],$$

where $I(\xi)=\int f^{\xi}(x)\,dx, \xi>0$ *and* $\xi\neq 1$. For the BMo density function (8), we have

$$I(\xi)=\frac{(\sqrt{2\pi})^{-\xi}}{\left[\Gamma\left(\frac{1}{2}\right)\right]^{\xi(b-1)}[B(a,b)]^{\xi}}\times$$

$$\int_{-\infty}^{\infty} e^{\{-\frac{\xi}{2}[x+\exp(-x)]\}} \left\{1-\frac{\gamma\left[\frac{1}{2},\frac{\exp(-x)}{2}\right]}{\Gamma\left(\frac{1}{2}\right)}\right\}^{\xi(a-1)} \left\{\gamma\left[\frac{1}{2},\frac{\exp(-x)}{2}\right]\right\}^{\xi(b-1)} dx. \tag{21}$$

Using the series expansion (9) in (??), we obtain

$$I(\xi)=\frac{(\sqrt{2\pi})^{-\xi}}{\left[\Gamma\left(\frac{1}{2}\right)\right]^{\xi(b-1)}[B(a,b)]^{\xi}} \sum_{j_1=0}^{\infty} \frac{(-1)^{j_1}\,\Gamma(\xi(a-1)+1)}{j_1!\,\Gamma(\xi(a-1)+1-j_1)\left[\Gamma\left(\frac{1}{2}\right)\right]^{j_1}}$$

$$\times\int_{-\infty}^{\infty}\exp\left\{-\frac{\xi}{2}[x+\exp(-x)]\right\}\left\{\gamma\left[\frac{1}{2},\frac{\exp(-x)}{2}\right]\right\}^{\xi(b-1)+j_1} dx.$$

Using (9) again and then applying the binomial expansion yields

$$I(\xi)=\frac{(\sqrt{2\pi})^{-\xi}}{\left[\Gamma\left(\frac{1}{2}\right)\right]^{\xi(b-1)}[B(a,b)]^{\xi}} \sum_{j_1,k_1=0}^{\infty}\sum_{r_1=0}^{k_1} \frac{(-1)^{j_1+k_1+r_1}\binom{k_1}{r_1}\Gamma(\xi(a-1)+1)\,\Gamma(\xi(b-1)+j_1+1)}{j_1!\,k_1!\,\Gamma(\xi(a-1)+1-j_1)\,\Gamma(\xi(b-1)+j_1+1-k_1)\left[\Gamma\left(\frac{1}{2}\right)\right]^{j_1}}$$

$$\times\int_{-\infty}^{\infty}\exp\{-\frac{\xi}{2}[x+\exp(-x)]\}\left\{\gamma\left[\frac{1}{2},\frac{\exp(-x)}{2}\right]\right\}^{r_1} dx.$$

Setting $u=\mathrm{e}^{-x}/2$, $I(\xi)$ reduces to

$$I(\xi)=\frac{(\sqrt{2\pi})^{-\xi}}{\left[\Gamma\left(\frac{1}{2}\right)\right]^{\xi(b-1)}[B(a,b)]^{\xi}} \sum_{j_1,k_1=0}^{\infty}\sum_{r_1=0}^{k_1} \frac{(-1)^{j_1+k_1+r_1}\binom{k_1}{r_1}\Gamma(\xi(a-1)+1)\,\Gamma(\xi(b-1)+j_1+1)}{j_1!\,k_1!\,\Gamma(\xi(a-1)+1-j_1)\,\Gamma(\xi(b-1)+j_1+1-k_1)\left[\Gamma\left(\frac{1}{2}\right)\right]^{j_1}}$$

$$\times\int_{0}^{\infty}u^{\frac{\xi}{2}-1}\exp(-\xi u)\left[\gamma\left(\frac{1}{2},u\right)\right]^{r_1} du.$$

Following similar developments in Theorem 1, we have

$$I(\xi)=\frac{(\sqrt{\pi})^{-\xi}}{\left[\Gamma\left(\frac{1}{2}\right)\right]^{\xi(b-1)}[B(a,b)]^{\xi}}$$

$$\times\sum_{j_1,k_1,m_1=0}^{\infty}\sum_{r_1=0}^{k_1} \frac{c_{m_1,r_1}(-1)^{j_1+k_1+r_1}\binom{k_1}{r_1}\Gamma(\xi(a-1)+1)\,\Gamma(\xi(b-1)+j_1+1)}{j_1!\,k_1!\,\Gamma(\xi(a-1)+1-j_1)\,\Gamma(\xi(b-1)+j_1+1-k_1)\left[\Gamma\left(\frac{1}{2}\right)\right]^{j_1}}$$

$$\times\int_{0}^{\infty}u^{m_1+\frac{\xi+r_1}{2}-1}\exp(-\xi u)\,du, \tag{22}$$

where c_{m_1,r_1} is defined in Section 3.1. The integral in equation (22) can be easily calculated from the result given by

Prudnikov *et al.* (1986, Vol.1, Section 2.3.3, integral 1). Hence,

$$I(\xi)=\frac{(\sqrt{\pi})^{-\xi}}{\left[\Gamma\left(\frac{1}{2}\right)\right]^{\xi(b-1)}[B(a,b)]^{\xi}}$$

$$\times\sum_{j_1,k_1,m_1=0}^{\infty}\sum_{r_1=0}^{k_1}\frac{c_{m_1,r_1}\,(-1)^{j_1+k_1+r_1}\binom{k_1}{r_1}\,\Gamma(\xi(a-1)+1)\,\Gamma(\xi(b-1)+j_1+1)}{j_1!\,k_1!\,\Gamma(\xi(a-1)+1-j_1)\,\Gamma(\xi(b-1)+j_1+1-k_1)\left[\Gamma\left(\frac{1}{2}\right)\right]^{j_1}}$$

$$\times\,\xi^{-\left(m_1+\frac{\xi+r_1}{2}\right)}\Gamma\left(m_1+\frac{\xi+r_1}{2}\right).$$

Finally, the Rényi entropy can be expressed as

$$j_R(\xi)=(1-\xi)^{-1}\Bigg\{-\xi\log(\sqrt{\pi})-\xi(b-1)\log\left[\Gamma\left(\frac{1}{2}\right)\right]-\xi\log\left[B(a,b)\right]$$

$$+\log\left\{\sum_{j_1,k_1,m_1=0}^{\infty}\sum_{r_1=0}^{k_1}\frac{c_{m_1,r_1}\,(-1)^{j_1+k_1+r_1}\binom{k_1}{r_1}\,\Gamma(\xi(a-1)+1)\;\Gamma(\xi(b-1)+j_1+1)}{j_1!\,k_1!\,\Gamma(\xi(a-1)+1-j_1)\;\Gamma(\xi(b-1)+j_1+1-k_1)\;\left[\Gamma\left(\frac{1}{2}\right)\right]^{j_1}}\right\}$$

$$+\left(m_1+\frac{\xi+r_1}{2}\right)\log(\xi)+\log\left[\Gamma\left(m_1+\frac{\xi+r_1}{2}\right)\right]\Bigg\}. \tag{23}$$

4. EXPANSIONS FOR THE ORDER STATISTICS

Moments of order statistics play an important role in quality control testing and reliability, where a practitioner needs to predict the failure of future items based on the times of a few early failures. These predictors are often based on moments of order statistics. We now derive an explicit expression for the density of the i th order statistic $X_{i:n}$, say $f_{i:n}(x)$, in a random sample of size n from the BMo distribution. It is well-known that

$$f_{i:n}(x)=\frac{f(x)}{B(i,n-i+1)}F(x)^{i-1}\{1-F(x)\}^{n-i},$$

for $i=1,\ldots,n$. Inserting (1) and (2) in the above equation, $f_{i:n}(x)$ can be written as

$$f_{i:n}(x)=\frac{n!\,g(x)}{(i-1)!\,(n-i)!\,B(a,b)}G(x)^{a-1}[1-G(x)]^{b-1}\left[I_{G(x)}(a,b)\right]^{i-1}\left[1-I_{G(x)}(a,b)\right]^{n-i}.$$

Substituting (6) and (7) in the last equation, the density $f_{i:n}(x)$ for $b>0$ real non-integer becomes

$$f_{i:n}(x)=\sum_{k=0}^{n-i}\frac{(-1)^k\genfrac{}{}{0pt}{}{n-i}{k}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}[x+\exp(-x)]\right\}\left\{1-\frac{\gamma\left\{\frac{1}{2},\frac{1}{2}\exp(-x)\right\}}{\Gamma(\frac{1}{2})}\right\}^{a(i+k)-1}}{\Gamma(\frac{1}{2})^{(b-1)}\;[\Gamma(b)^{i+k-1}]^{-1}\;B(a,b)^{i+k}\;B(i,n-1+i)\left\{\gamma\left\{\frac{1}{2},\frac{1}{2}\exp(-x)\right\}\right\}^{-(b-1)}}$$

$$\times \left[\sum_{j=0}^{\infty} \frac{(-1)^j \left\{ 1 - \frac{\gamma\left\{\frac{1}{2}, \frac{1}{2}\exp(-x)\right\}}{\Gamma(\frac{1}{2})} \right\}^j}{(a+j)\; j!\,\Gamma(b-j)} \right]^{i+k-1} . \tag{24}$$

We define

$$A = \sum_{j=0}^{\infty} \frac{(-1)^j \left\{ 1 - \frac{\gamma\left\{\frac{1}{2}, \frac{1}{2}\exp(-x)\right\}}{\Gamma(\frac{1}{2})} \right\}^j}{(a+j)\; j!\,\Gamma(b-j)} .$$

Setting $u = \mathrm{e}^{-x}/2$ and using the series expansion

$$1 - \frac{\gamma(\alpha, x)}{\Gamma(\alpha)} = x^{\alpha-1} \exp(-x) \sum_{m=0}^{\infty} \frac{x^{-m}}{\Gamma(\alpha - m)},$$

the quantity A becomes

$$A = \sum_{j=0}^{\infty} \left\{ \frac{(-1)^j\; u^{-\frac{j}{2}}\; \exp(-uj) \left[\sum_{m=0}^{\infty} \frac{u^{-m}}{\Gamma(\frac{1}{2} - m)} \right]^j}{(a+j)\; j!\,\Gamma(b-j)} \right\}.$$

Hence,

$$A = \sum_{j=0}^{\infty} \left\{ \frac{(-1)^j\; u^{-\frac{3j}{2}}\; \exp(-uj) \left[\sum_{m_1=0}^{\infty} \cdots \sum_{m_j=0}^{\infty} \frac{u^{-(m_1+\cdots+m_j)}}{\Gamma(\frac{1}{2} - m_1) \ldots \Gamma(\frac{1}{2} - m_j)} \right]}{(a+j)\; j!\,\Gamma(b-j)} u^j \right\}. \tag{25}$$

Again we use the identity $(\sum_{k=0}^{\infty} a_k\; x^k)^n = \sum_{k=0}^{\infty} c_{k,n}\; x^k$ (see Gradshteyn and Ryzhik, 2000), where a_k now comes by identifying (25) with the corresponding quantity which is elevated to the power $i+k-1$ in equation (24). We have

$$a_k = \frac{(-1)^k\; u^{-\frac{3k}{2}}\; \exp(-uk) \left[\sum_{m_1=0}^{\infty} \cdots \sum_{m_k=0}^{\infty} \frac{u^{-(m_1+\cdots+m_k)}}{\Gamma(\frac{1}{2} - m_1) \ldots \Gamma(\frac{1}{2} - m_k)} \right]}{(a+k)\; k!\,\Gamma(b-k)},$$

$$c_{0,n} = a_0^n \qquad \textit{and} \qquad c_{k,n} = (ka_0)^{-1}\sum_{l=1}^{k}(nl-k+l)a_l\, c_{k-l,n}$$

for $k = 1,2,\ldots$ Thus, after some algebra, we obtain

$$f_{i:n}(x) = \sum_{k=0}^{n-i}\sum_{j=0}^{\infty}\frac{(-1)^k \binom{n-i}{k} \Gamma(b)^{i+k-1}\, B[a(i+k)+j,b]\, d_{i,j,k}}{B(a,b)^{i+k}\; B(i,n-1+i)}\, f_{i,j,k}(x), \tag{26}$$

where

$$f_{i,j,k}(x) = \frac{\exp\{-\frac{1}{2}[x+\exp(-x)]\}}{\sqrt{2\pi}\,\Gamma\left(\frac{1}{2}\right)^{b-1}\; B[a(i+k)+j,b]}\left\{1-\frac{\gamma\left\{\frac{1}{2},\frac{1}{2}\exp(-x)\right\}}{\Gamma(\frac{1}{2})}\right\}^{a(i+k)+j-1} \times \left\{\gamma\left\{\frac{1}{2},\frac{1}{2}\exp(-x)\right\}\right\}^{b-1}$$

denotes the density of the BMo $(a(i+k)+j,b,0,1)$ distribution and the constants $d_{i,j,k}$ can be obtained recursively from the following equations

$$d_{i,0,k} = \left\{\frac{1}{a\Gamma(b)}\right\}^{i+k-1} \quad \textit{and} \quad d_{i,j,k} = \frac{a\Gamma(b)}{j}\sum_{l=1}^{j}\frac{(-1)^l\,\{l(i+k)-j\}}{(a+l)\; l!\,\Gamma(b-l)}\, c_{j-l,i+k-1},\; j \geq 1.$$

The density function of the BMo order statistics is then an infinite linear combination of BMo density functions. Hence, the ordinary and central moments of the order statistics can be calculated directly from those quantities of the proposed distribution given in Section 3.1. For $b>0$ integer, expansion (26) holds but the sum in j stops at $(b-1)(k+i-1)$. Analogously, the generating function of the standard BMo order statistics can be determined from the result in Section 3.2.

An alternative expansion for the density of the order statistics can follow from the identity $(\sum_{i=0}^{\infty} a_i)^k = \sum_{m_1,\ldots,m_k=0}^{\infty} a_{m_1}\ldots a_{m_k}$ for k a positive integer. Using this identity and equation (24), for $b>0$ real non-integer, it turns out that

$$f_{i:n}(x) = \sum_{k=0}^{n-i}\sum_{m_1=0}^{\infty}\cdots\sum_{m_{i+k-1}=0}^{\infty}\frac{(-1)^k \binom{n-i}{k} \Gamma(b)^{i+k-1}\frac{1}{\sqrt{2\pi}}\exp\{-\frac{1}{2}[x+\exp(-x)]\}\left\{\gamma\left\{\frac{1}{2},\frac{1}{2}\exp(-x)\right\}\right\}^{(b-1)}}{\Gamma(\frac{1}{2})^{(b-1)}\; B(i,n-1+i)\; B(a,b)^{i+k}}$$

$$\times\frac{(-1)^{k+\sum_{j=1}^{i+k-1}m_j}\binom{n-i}{k}\Gamma(b)^{i+k-1}\left\{1-\frac{\gamma\left\{\frac{1}{2},\frac{1}{2}\exp(-x)\right\}}{\Gamma(\frac{1}{2})}\right\}^{a(i+k)+\sum_{j=1}^{i+k-1}m_j}}{\prod_{j=1}^{i+k-1}(a+m_j)\; m_j!\,\Gamma(b-m_j)}.$$

Hence,

$$f_{i:n}(x) = \sum_{k=0}^{n-i} \sum_{m_1=0}^{\infty} \dots \sum_{m_{i+k-1}=0}^{\infty} \delta_{i,k}\, f_{i,k}(x), \tag{27}$$

where

$$f_{i,k}(x) = \frac{\exp\{-\frac{1}{2}[x+\exp(-x)]\}}{\sqrt{2\pi}\,\Gamma\left(\frac{1}{2}\right)^{b-1} B[a(i+k)+\sum_{j=1}^{i+k-1} m_j, b]} \times$$

$$\left\{1 - \frac{\gamma\left\{\frac{1}{2}, \frac{1}{2}\exp(-x)\right\}}{\Gamma(\frac{1}{2})}\right\}^{a(i+k)+\sum_{j=1}^{i+k-1} m_j - 1} \left\{\gamma\left\{\frac{1}{2}, \frac{1}{2}\exp(-x)\right\}\right\}^{b-1}$$

denotes the density of the BMo $(a(i+k)+\sum_{j=1}^{i+k-1} m_j, b, 0, 1)$ distribution and

$$\delta_{i,k} = \frac{(-1)^{k+\sum_{j=1}^{i+k-1} m_j} \binom{n-i}{k} \Gamma(b)^{i+k-1}\, B(a(i+k)+\sum_{j=1}^{i+k-1} m_j, b)}{B(a,b)^{i+k}\, B(i, n-i+1) \prod_{j=1}^{i+k-1} (a+m_j)\, m_j!\, \Gamma(b-m_j)}.$$

The constants $\delta_{i,k}$ are easily obtained given i, n, k and a sequence of indices $m_1, \dots, m_{i+k-1}$. The sums in (27) extend over all $(i+k)$-tuples ($k, m_1, \dots, m_{i+k-1}$) of non-negative integers and can be implemented in a computer language (such as Mathematica) using just a few lines of code. If $b > 0$ is an integer, equation (27) holds but the indices $m_1, \dots, m_{i+k-1}$ vary from zero to $b-1$. Expansion (26) is much simpler to be calculated numerically in applications and the corresponding CPU times are usually smaller than those from (27).
The s th moment of $X_{i:n}$ for $b > 0$ real non-integer comes from (26)

$$E(X_{i:n}^s) = \sum_{k=0}^{n-i} \sum_{j=0}^{\infty} \frac{(-1)^k \binom{n-i}{k} \Gamma(b)^{i+k-1}\, B(a(i+k)+j, b)\, d_{i,j,k}}{B(i, n-i+1)\, B(a,b)^{i+k}} E(X_{i,j,k}^s), \tag{28}$$

where $X_{i,j,k} : BMo(a(i+k)+j, b, 0, 1)$ and the constants $d_{i,j,k}$ were defined before. If b is an integer, the sum in j stops at $b-1$.
From equation (27), we can obtain an alternative expression for the moments of the order statistics valid for $b > 0$ real non-integer

$$E(X_{i:n}^s) = \sum_{k=0}^{n-i} \sum_{m_1=0}^{\infty} \dots \sum_{m_{i+k-1}=0}^{\infty} \delta_{i,k}\, E(X_{i,k}^s), \tag{29}$$

where $X_{i,k} : BMo(a(i+k)+\sum_{j=1}^{i+k-1} m_j, b, 0, 1)$. If $b > 0$ is an integer, the indices $m_1, \dots, m_{i+k-1}$ stop at $b-1$.
We therefore offer two alternative expressions (28) and (29) for the moments of the BMo order statistics, which are the main results of this section.
From (28) and (29), we can easily derive expansions for the L-moments (Hosking, 1990) of the BMo distribution as linear functions of expected order statistics given by

$$\lambda_{r+1} = r(r+1)^{-1}\sum_{k=0}^{r}\frac{(-1)^k}{k}E(X_{r+1-k:r+1}), r = 0,1,\ldots$$

The first four L-moments are: $\lambda_1 = E(X_{1:1})$, $\lambda_2 = \frac{1}{2}E(X_{2:2} - X_{1:2})$, $\lambda_3 = \frac{1}{3}E(X_{3:3} - 2X_{2:3} + X_{1:3})$ and $\lambda_4 = \frac{1}{4}E(X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4})$.

5. ESTIMATION AND INFERENCE

The parameters of the BMo distribution are estimated by the method of maximum likelihood. If X has the BMo distribution with vector of parameters $\lambda = (a,b,\mu,\sigma)^{\mathrm{T}}$, the log-likelihood for the model parameters from a single observation x of X is given by

$$\ell(\lambda) = \log\left(\frac{1}{\sqrt{2\pi}}\right) - \log(\sigma) - \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right) - \frac{1}{2}\exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right] - \log[B(a,b)]$$

$$-\ (b-1)\log\left[\Gamma\left(\frac{1}{2}\right)\right] + (a-1)\log\left\{1 - \frac{\gamma\left\{\frac{1}{2},\frac{1}{2}\exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right]\right\}}{\Gamma(\frac{1}{2})}\right\}$$

$$+\ (b-1)\log\left\{\gamma\left\{\frac{1}{2},\frac{1}{2}\exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right]\right\}\right\}, \quad -\infty < x < \infty.$$

The components of the unit score vector $U = \left(\frac{\partial\ell}{\partial a},\frac{\partial\ell}{\partial b},\frac{\partial\ell}{\partial\mu},\frac{\partial\ell}{\partial\sigma}\right)^{\mathrm{T}}$ are

$$\frac{\partial\ell}{\partial a} = \log\left\{1 - \frac{\gamma\left[\frac{1}{2},\frac{1}{2}\exp(-z)\right]}{\Gamma(\frac{1}{2})}\right\} - \psi(a) + \psi(a+b),$$

$$\frac{\partial\ell}{\partial b} = \log\left\{\gamma\left[\frac{1}{2},\frac{1}{2}\exp(-z)\right]\right\} - \log\left[\Gamma\left(\frac{1}{2}\right)\right] - \psi(b) + \psi(a+b),$$

$$\frac{\partial\ell}{\partial\mu} = \frac{1}{2\sigma} - \frac{1}{2\sigma}\exp(-z) + (a-1)\left\{\frac{\frac{1}{2\sigma}\Gamma\left(\frac{1}{2}\right)(\sqrt{2\pi}\sigma)^{-1}\exp[-z-\exp(-z)][1+\exp(-z)]}{\Gamma\left(\frac{1}{2}\right) - \gamma\left[\frac{1}{2},\frac{1}{2}\exp(-z)\right]}\right\}$$

$$+(b-1)\left\{\frac{\frac{\sqrt{2}}{2\sigma}\exp\left[-\frac{1}{2}\exp(-z)\right]}{\gamma\left[\frac{1}{2},\frac{1}{2}\exp(-z)\right]}\right\},$$

$$\frac{\partial \ell}{\partial \sigma} = -\frac{1}{\sigma} + \frac{z}{2\sigma} - \frac{z}{2\sigma}\exp(-z) + (b-1)\left\{ \frac{\frac{z\sqrt{2}}{2\sigma}\exp\left\{-\frac{1}{2}[z+\exp(-z)]\right\}}{\gamma\left[\frac{1}{2},\frac{1}{2}\exp(-z)\right]} \right\}$$

$$+(a-1)\left\{ \frac{\Gamma\left(\frac{1}{2}\right)(\sqrt{2\pi\sigma^2})^{-1}\exp\left\{-\frac{1}{2}[z+\exp(-z)]\right\}\left[\frac{z}{2}+z\exp(-z)-1\right]}{\Gamma\left(\frac{1}{2}\right)-\gamma\left[\frac{1}{2},\frac{1}{2}\exp(-z)\right]} \right\},$$

where $z=(x-\mu)/\sigma$ and $\psi(.)$ is the digamma function.

For a random sample $x=(x_1,\ldots,x_n)^{\mathrm{T}}$ of size n from X, the total log-likelihood is $\ell_n=\ell_n(\lambda)=\sum_{i=1}^{n}\ell^{(i)}(\lambda)$, where $\ell^{(i)}(\lambda)$ is the log-likelihood for the ith observation $(i=1,\ldots,n)$. The total score function is $U_n=\sum_{i=1}^{n}U^{(i)}$, where $U^{(i)}$ has the previous form for $i=1,\ldots,n$. The maximum likelihood estimate (MLE) λ of λ is the solution of the nonlinear system of equations $U_n=0$.

For interval estimation and tests of hypotheses on the parameters in λ, we require the 4×4 unit expected information matrix

$$K=K(\lambda)=\begin{pmatrix} \kappa_{aa} & \kappa_{a,b} & \kappa_{a,\mu} & \kappa_{a,\sigma} \\ . & \kappa_{b,b} & \kappa_{b,\mu} & \kappa_{b,\sigma} \\ . & . & \kappa_{\mu,\mu} & \kappa_{\mu,\sigma} \\ . & . & . & \kappa_{\sigma,\sigma} \end{pmatrix},$$

whose elements are given in Appendix A.

Under conditions that are fulfilled for parameters in the interior of the parameter space, the asymptotic distribution of $\sqrt{n}(\lambda-\lambda)$ is $N_4(0,K(\lambda)^{-1})$. The estimated asymptotic distribution $N_4(0,n^{-1}K(\lambda)^{-1})$ of λ can be used to construct approximate confidence intervals for the parameters and for the hazard rate and survival functions. An asymptotic confidence interval with significance level γ for each parameter λ_r is given by

$$ACI(\lambda_r,100(1-\gamma)\%)=(\hat{\lambda}_r-z_{\gamma/2}\sqrt{\hat{\kappa}^{\lambda_r,\lambda_r}},\hat{\lambda}_r+z_{\gamma/2}\sqrt{\hat{\kappa}^{\lambda_r,\lambda_r}}),$$

where $\hat{\kappa}^{\lambda_r,\lambda_r}$ is the rth diagonal element of $n^{-1}K(\lambda)^{-1}$ estimated at λ, for $r=1,\cdots,4$, and $z_{\gamma/2}$ is the quantile $1-\gamma/2$ of the standard normal distribution.

The likelihood ratio (LR) statistic is useful for testing goodness-of-fit of the BMo distribution and for comparing this distribution with some of its special sub-models. We can compute the maximum values of the unrestricted and restricted log-likelihoods to construct LR statistics for testing some sub-models of the BMo distribution. For example, we may use the LR statistic to check if the fit using the BMo distribution is statistically ``superior" to a fit using the Moyal distribution for a given data set. In any case, considering the partition $\lambda=(\lambda_1^{\mathrm{T}},\lambda_2^{\mathrm{T}})^{\mathrm{T}}$, tests of hypotheses of the type $H_0:\lambda_1=\lambda_1^{(0)}$ versus $H_A:\lambda_1\neq\lambda_1^{(0)}$ can be performed via the LR statistic $w=2\{\ell(\lambda)-\ell(\lambda)\}$, where λ and λ are the MLEs of λ under H_A and H_0, respectively. Under the null hypothesis H_0, $w\xrightarrow{d}\chi_q^2$, where q is the dimension of the vector λ_1 of interest. The LR test rejects H_0 if $w>\xi_\gamma$, where ξ_γ denotes the upper $100\gamma\%$ point of the χ_q^2 distribution. From the score vector and the information matrix given before, we can also construct score and Wald statistics that are asymptotically equivalent to LR statistics.

6. APPLICATIONS

In this section, we use several real data sets to compare the fits of the BMo distribution with those of the beta normal, skew-normal and Moyal distributions. In each case, the parameters are estimated by maximum likelihood as described in Section 5 using the subroutine NLMixed in SAS. First, we describe the data sets. Then, we provide the MLEs (and the corresponding standard errors in parentheses) of the model parameters and the values of the AIC (Akaike Information Criterion), CAIC (Consistent Akaike Information Criterion) and BIC (Bayesian Information Criterion) statistics. The lower the values of these statistics, the better the fit. Next, we perform the LR tests (Section 5). Finally, the histograms of these data sets are provided for a visual comparison of the fitted density functions.

(i) The wheaton river data

As a first example, we consider the data set (Akinsete *et al.*, 2008) on the exceedances of flood peaks (in m3/s) of the Wheaton River near Carcross in Yukon Territory, Canada. The data consist of 72 exceedances for the years 1958-1984, rounded to one decimal place. These data were analyzed by Choulakian and Stephens (2001).

(ii) Tubercle bacilli data

The data, originally reported by Bjerkedal (1960), represent the survival times of guinea pigs injected with different doses of tubercle bacilli. These data were analyzed by Kundu *et al.* (2008) and Leiva *et al.* (2009). It is known that guinea pigs have high susceptibility to human tuberculosis and that they were used in this study. Here, we are primarily concerned with the animals in the same cage that were under the same regimen. The regimen number is the common logarithm of the number of bacillary units in 0.5 ml of challenge solution, that is, regimen 6.6 corresponding to 4.0×106 bacillary units per 0.5 ml $(\log(4.0\times 106)=6.6)$.

(iii) Air pollution data

To obtain the level of air pollution and its associated adverse effects on humans in Santiago, Chile, the National Commission of Environment (CONAMA) of the government of Chile collects data on sulfur dioxide (SO_2) concentrations in the air. The data correspond to the hourly SO_2 concentrations (in ppb, American parts per billion, ppm $\times 1000$) observed at a monitoring station located in Santiago city. These data were analyzed by Balakrishnan *et al.* (2009) and Leiva *et al.* (2009).

Table 1: Descriptive statistics.

Data	Mean	Median	Mode	SD	Variance	Skewness	Kurtosis	Min.	Max.
Wheaton river	12.2	9.5	1.7	12.3	151.2	1.5	3.2	0.1	64
Tubercle bacilli	99.8	70.0	60.0	81.1	6580.1	1.8	2.9	12	376
Air pollution	2.9	3.0	2.0	1.9	3.5	4.6	40.7	1.0	25

Table 0 gives a descriptive summary of each sample. The wheaton river, tubercle bacilli and air pollution data have positive skewness and kurtosis, larger values of these sample moments are shown in the tubercle bacilli data.
We now compute the MLEs and the AIC, BIC and CAIC information criteria for the fitted models in each data set. The classical estimates of μ and σ for the normal distribution are taken as starting values for the fits of the BMo, Moyal, beta normal and skew-normal distributions. The results are reported in Table 1. In any case, since the values of the three statistics are smaller for the BMo distribution compared to those values of the Moyal, beta normal and skew-normal distributions, we conclude that the new distribution is a very competitive model for data analysis.

Table 2: MLEs and information criteria.

Wheaton river	a	b	μ	σ	AIC	CAIC	BIC
BMo	0.2693	0.2612	5.5967	2.0295	257.5	258.1	266.6
	(0.0370)	(0.0479)	(0.4095)	(0.2230)			
Moyal	1	1	5.4092	4.8127	271.6	271.8	276.1
	-	-	(0.8569)	(0.5126)			
Beta normal	141.14	134.0	6.9376	161.35	572.6	573.2	581.7
	(0.4302)	(0.3954)	(1.8309)	(13.4563)			
		λ	μ	σ			
Skew-normal	-	-0.0026	12.2279	12.2125	570.7	571.0	577.5
	-	(4.3820)	(2.7227)	(1.0214)			
Tubercle bacilli	a	b	μ	σ	AIC	CAIC	BIC
BMo	0.4841	0.4568	61.0070	17.7372	787.9	788.5	797.0
	(0.0953)	(0.0928)	(7.1896)	(2.3706)			
Moyal	1	1	59.8823	28.0211	790.9	791.1	795.4
	-	-	(4.9499)	(2.8541)			
Beta normal	1.7519	0.2313	16.0616	44.4357	827.8	828.4	836.9
	(0.7558)	(0.0303)	(17.3197)	(2.3460)			
		λ	μ	σ			
Skew-normal	-	19.9240	15.4379	116.66	800.5	800.9	807.3
	-	(13.4867)	(6.6814)	(10.8566)			
Air pollution	a	b	μ	σ	AIC	CAIC	BIC
BMo	0.2488	0.9009	3.4512	0.7542	2858.3	2858.4	2877.3
	(0.0144)	(0.1164)	(0.2455)	(0.0758)			
Moyal	1	1	2.0691	0.6708	2872.7	2872.8	2882.2
	-	-	(0.0350)	(0.0188)			
Beta normal	3.2438	1.0887	0.5395	2.9670	3433.0	3433.1	3452.0
	(0.3952)	(0.0498)	(0.3064)	(0.0648)			
		λ	μ	σ			
Skew-normal	-	7.0336	1.0747	2.6400	3081.0	3081.1	3095.3
	-	(1.0518)	(0.0593)	(0.0761)			

A formal test for the third skewness parameter in the BMo distribution can be based on the LR statistics (Section 5). Applying this test to the three data sets, the results are shown in Table 2. For the three data sets, we reject the null hypothesis $H_0 : a = b = 1$ in favor of the BMo distribution. This fact provides an evidence of the importance for the three skewness parameters when modeling real data.

Table 3: LR tests.

Wheaton river	Hypotheses	Statistic w	p -value
BMo vs Moyal	$H_0 : a = b = 1$ vs $H_1 : H_0$ *is false*	18.1	0.0001
Tubercle bacilli	Hypotheses	Statistic w	p -value
BMo vs Moyal	$H_0 : a = b = 1$ vs $H_1 : H_0$ *is false*	7.0	0.0302
Air pollution	Hypotheses	Statistic w	p -value
BMo vs Moyal	$H_0 : a = b = 1$ vs $H_1 : H_0$ *is false*	20.4	0.00004

The histogram of the data and the plots of the fitted BMo, Moyal, beta normal and skew-normal distributions are given in Figures 3, 4 and 5. These plots show some evidence that the BMo distribution seems superior to the other distributions in terms of model fit.

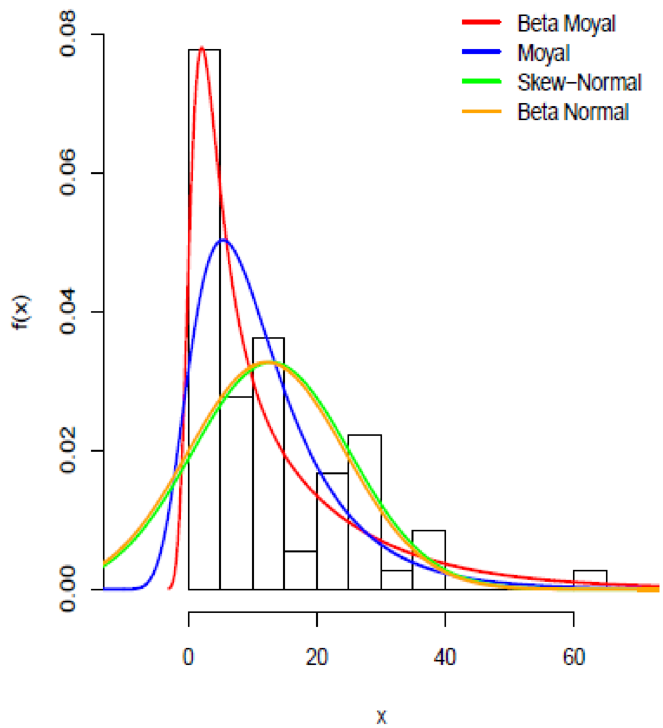


Figure 4: Estimated densities of the BMo, Moyal, beta normal and skew-normal models for the wheaton river data.

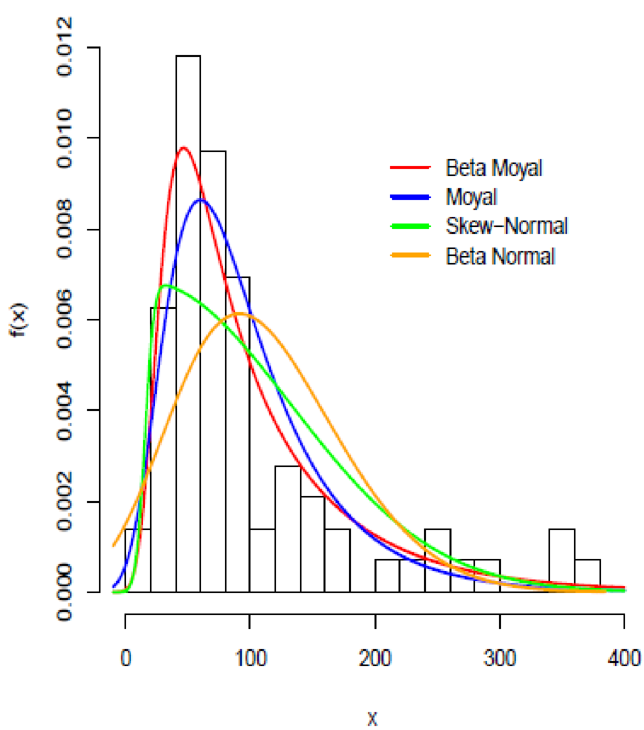


Figure 5: Estimated densities of the BMo, Moyal, beta normal and skew-normal models for the tubercle bacilli data.

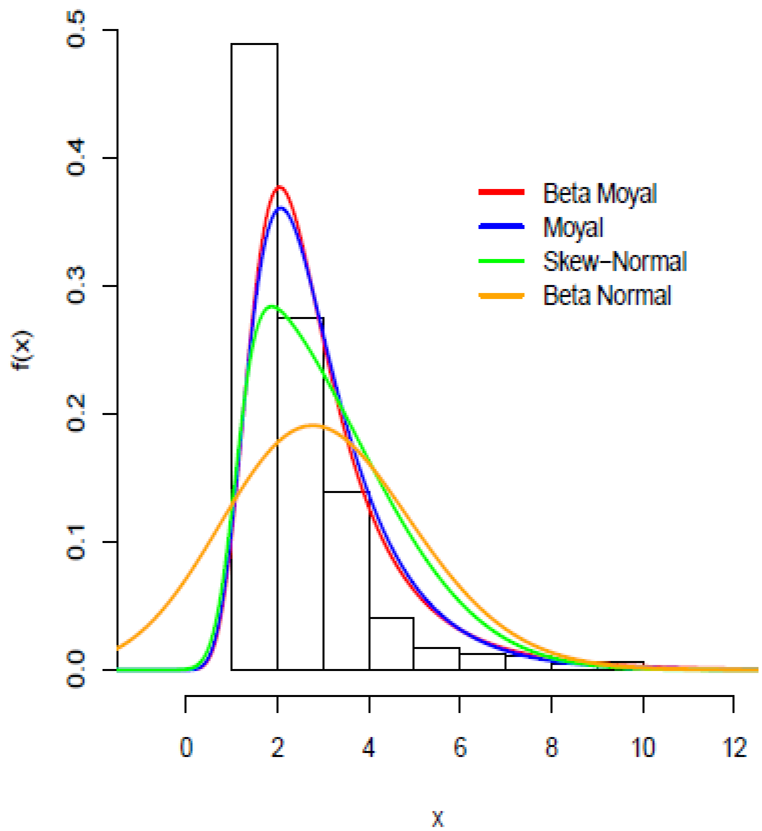


Figure 6: Estimated densities of the BMo, Moyal, beta normal and skew-normal models for the air pollution data.

7. CONCLUSIONS

In this article, we propose a new model called the beta Moyal (BMo) distribution to extend the Moyal distribution in the analysis of skew data with real support. An obvious reason for generalizing a ``standard distribution" is because the generalized form provides greater flexibility in modeling real data. We provide a mathematical treatment of the new distribution including expansions for its distribution and density functions. We derive expansions for the moments, generating function, mean deviations and moments of order statistics. The estimation of parameters is performed by the method of maximum likelihood and the information matrix is derived. We adopt the likelihood ratio (LR) statistic to compare the new model with its baseline model. Three applications of the BMo distribution to real data show that the new distribution can be used quite effectively to provide better fits than the beta normal, Moyal and skew-normal distributions.

Appendix A

The elements of the 4×4 unit expected information matrix are given by

$$\kappa_{\sigma,\sigma} = -\frac{1}{\sigma} - \frac{1}{2\sigma^2}\left[T_{0,0,2,0,1,0,0} - T_{0,0,2,0,2,0,0}\right] - \frac{\sqrt{2}(a-1)}{\sigma^3\sqrt{\pi}}\left\{\frac{2}{\sqrt{\pi}}\left[(1+\log(2))\ T_{1,0,1,1,0,0,0}\right.\right.$$

$$+\ \log(4)\ T_{1,0,2,1,0,0,0} - 4\ T_{1,0,3,1,0,1,0} + T_{1,0,1,1,0,1,0}\Big] + \frac{1}{2\sqrt{\pi}}\left[\frac{1}{2}\ T_{1,0,1,1,2,0,0}\right.$$

$$+\ T_{1,0,3,1,2,0,0} - 4\ T_{1,0,5,1,2,0,0} - 2\ T_{1,0,3,1,1,0,0} - 4\ T_{1,0,3,1,1,0,1} + 2\ T_{1,0,2,2,2,0,0}$$

$$\left.\left. +\ 8\ T_{1,0,4,2,2,0,0}\right] + \frac{2}{\pi}\ T_{2,0,2,2,1,0,0}\right\} - \sqrt{2}(b-1)\left\{\frac{1}{\sqrt{\pi}}\ T_{0,1,1,1,1,0,0}\right.$$

$$+\frac{1}{4\sqrt{\pi}\sigma^2}\left[\sqrt{2}\ T_{0,1,1,1,2,0,0}-2\ T_{0,1,3,1,2,0,0}\right]-\frac{\sqrt{2}}{2\pi\sigma^2}\ T_{0,2,2,2,2,0,0}\Bigg\},$$

$$\kappa_{\mu,\sigma}=\frac{1}{2\sigma^2}-\frac{1}{\sigma^2}\left[T_{0,0,2,0,0,0,0}+T_{0,0,2,0,1,0,0}\right]-\frac{(a-1)}{\sigma^3}\Bigg\{\frac{1}{\sqrt{\pi}}\Bigg[\frac{1}{4}\ T_{1,0,1,1,1,0,0}+\frac{3}{2}\ T_{1,0,3,1,1,0,0}$$
$$+3\ T_{1,0,3,1,0,0,0}-T_{1,0,1,1,0,0,0}\Bigg]+\frac{1}{2\pi}\left[T_{2,0,2,2,1,0,0}+4\ T_{2,0,4,2,1,0,0}+2\ T_{2,0,2,2,0,0,0}\right]\Bigg\}$$
$$+\frac{(b-1)}{\sigma^2}\left\{\frac{1}{\sqrt{\pi}}\ T_{0,1,1,1,0,0,0}-\frac{1}{\pi}\ T_{0,2,2,2,1,0,0}+\frac{1}{2\sqrt{\pi}}\left[T_{0,1,1,1,1,0,0}-2\ T_{0,1,3,1,1,0,0}\right]\right\},$$

$$\kappa_{b,\mu}=-\frac{1}{2\sigma}\sqrt{\frac{2}{\pi}}\ T_{0,1,0,1,0,0,0},\quad \kappa_{b,\sigma}=\frac{1}{\sigma\sqrt{\pi}}T_{0,1,1,1,1,0,0},\quad \kappa_{a,b}=-\psi'(a+b),$$

$$\kappa_{\mu,\mu}=\frac{1}{\sigma^2}T_{0,0,2,0,0,0,0}-\frac{(a-1)}{\sigma^3\sqrt{2}}\Bigg\{\frac{2}{\sqrt{\pi}}\left[\left(1-\frac{1}{\sigma}\right)T_{1,0,4,2,0,0,0}+\frac{2}{\sigma}\ T_{1,0,8,2,0,0,0}\right]+\frac{\sqrt{2}}{2\pi}\Big[T_{2,0,2,3,0,0,0}$$
$$+2\ T_{2,0,4,3,0,0,0}\Big]\Bigg\}+\frac{(b-1)\sqrt{2}}{4\sigma^2}\left[\frac{2}{\sqrt{\pi}}\ T_{0,1,2,1,0,0,0}+\frac{\sqrt{2}}{\pi}\ T_{0,2,0,2,0,0,0}\right],$$

$$\kappa_{a,\sigma}=\frac{1}{\pi\sigma^2}\left[(1+\log(2))\ T_{1,0,1,1,0,0,0}+\log(4)\ T_{1,0,2,1,0,0,0}+4\ T_{1,0,3,1,0,1,0}+T_{1,0,1,1,0,1,0}\right],$$

$$\kappa_{a,\mu}=-\frac{1}{\sigma^2\sqrt{2\pi}}\left[T_{1,0,2,2,0,0,0}+2\ T_{1,0,4,2,0,0,0}\right],\kappa_{a,a}=\psi'(a)-\psi'(a+b),\kappa_{b,b}=\psi'(b)-\psi'(a+b).$$

Here, we assume that a random variable V has a beta distribution with parameters a and b and define the expected value

$$T_{i,j,k,l,m,n,p}=E\Big\{V^{-i}(1-V)^{-j}\left[erf^{-1}(1-V)\right]^k\exp\left\{-l\left[erf^{-1}(1-V)\right]^2\right\}\left[\log\left\{2\left[erf^{-1}(1-V)\right]^2\right\}\right]^m$$
$$\times\left[\log\left\{erf^{-1}(1-V)\right\}\right]^n\left[\exp\left\{2\left[erf^{-1}(1-V)\right]^2\log\left\{2\left[erf^{-1}(1-V)\right]^2\right\}\right\}\right]^p\Big\}.$$

These expected values can be determined numerically using Maple and Mathematica for any a and b.
For example, for a = 2:5 and b = 3, we easily calculate all T's in the information matrix:
$T_{0;0;2;0;1;0;0}$ = 0:4598979; $T_{0;0;2;0;2;0;0}$ = 0:1271101; $T_{0;1;1;1;1;0;0}$ = ¡0:4559037; $T_{0;1;1;1;2;0;0}$ = 1:800166; $T_{0;1;3;1;2;0;0}$ = 0:1077819 ; $T_{0;2;2;2;2;0;0}$ = 0:9190356; $T_{0;0;2;0;0;0;0}$ = 0:4948074; $T_{0;1;1;1;0;0;0}$ = 0:6847316; $T_{0;2;2;2;1;0;0}$ = 0:291509; $T_{0;1;3;1;1;0;0}$ = ¡0:08361248; $T_{0;1;0;1;0;0;0}$ = 1:745719; $T_{0;1;2;1;0;0;0}$ = 0:4406842; $T_{0;2;0;2;0;0;0}$ = 3:955121; $T_{1;0;1;1;0;0;0}$ = 0:9004557; $T_{1;0;1;1;0;1;0}$ =; ¡0:8180274; $T_{1;0;1;1;2;0;0}$ = 0:3652951; $T_{1;0;1;1;1;0;0}$ = ¡0:2425763; $T_{1;0;2;1;0;0;0}$ = 0:39603; $T_{1;0;2;2;2;0;0}$ =; 0:04201742; $T_{1;0;2;2;0;0;0}$ = 0:01161524; $T_{1;0;3;1;0;1;0}$ = ¡0:2319999; $T_{1;0;3;1;2;0;0}$ = 0:2246422; $T_{1;0;3;1;1;0;0}$ = ¡0:2028589; $T_{1;0;3;1;1;0;1}$ = ¡0:1049516; $T_{1;0;3;1;0;0;0}$ = 0:7749216; $T_{1;0;5;1;2;0;0}$ = 0:004927196; $T_{1;0;4;2;2;0;0}$ = 0:003980552; $T_{1;0;4;2;0;0;0}$ = 0:09122807; $T_{2;0;2;2;1;0;0}$ = 1:632044; $T_{2;0;2;2;0;0;0}$ = 2:917779; $T_{2;0;2;3;0;0;0}$ = 2:567143; $T_{2;0;4;2;1;0;0}$ = ¡0:5019668; $T_{2;0;4;3;0;0;0}$ = 2:570722; $T_{1;0;8;2;0;0;0}$ = 0:08920794.

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